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Bounds for Solutions to a Class of Damped Integrodifferential Equations in Hilbert Space with Applications to the Theory of Nonconducting Material Dielectrics*

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Abstract

Let $T > 0$ be an arbitrary real number and H, H_+ real Hilbert spaces with $H_+ \subseteq H$ algebraically and topologically and H_+ dense in H. Let H_- be the dual of H_+ via the inner product of H and denote by $L_S(H_+,H_-)$ the space of symmetric bounded linear operators from H_+ into H_-. We prove that the evolution of the electric displacement field in a simple class of holohedral isotropic dielectrics can be modeled by an abstract initial-value problem of the form

$$\underset{\sim}{u}_{tt} - \alpha \underset{\sim}{u}_t - \underset{\sim}{L}\underset{\sim}{u} + \int_0^t \underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)d\tau = \beta(t)\underset{\sim}{u}_o, \quad 0 \le t < T$$

$$\underset{\sim}{u}(0) = \underset{\sim}{u}_o, \quad \underset{\sim}{u}_t(0) = \underset{\sim}{u}_1 \quad (\underset{\sim}{u}_o, \underset{\sim}{u}_1 \in H_+)$$

where $\underset{\sim}{L} \in L_S(H_+, H_-)$, $\underset{\sim}{M}(t) \in L^2([0,T); L_S(H_+, H_-))$, $\beta(t) \in C^1([0,T))$, and α is an arbitrary (non-zero) real number. By employing a logarithmic convexity argument we derive growth estimates for

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solution of the above system which lie in uniformly bounded classes of the form

$$\mathcal{N} = \{\underset{\sim}{u} \in C^2([0,T);\, H_+) \,\Big|\, \sup_{[0,T)} ||\underset{\sim}{u}||_{H_+} \leq N\}$$

for some $N > 0$; our results are derived under a variety of assumptions concerning α, $\beta(t)$, and the initial data (without making any definiteness assumptions on the operators $\underset{\sim}{L}$ or $\underset{\sim}{M}(t)$, $0 \leq t < T$) and are used to obtain growth estimates for the electric displacement field $\underset{\sim}{D}(\underset{\sim}{x},t)$ in rigid dielectrics which satisfy constitutive relations of the form

$$\underset{\sim}{D}(\underset{\sim}{x},t) = a_o\underset{\sim}{E}(\underset{\sim}{x},t) + \int_0^t \phi(t-\tau)\underset{\sim}{E}(\underset{\sim}{x},\tau)d\tau$$

$$\underset{\sim}{H}(\underset{\sim}{x},t) = b_o\underset{\sim}{B}(\underset{\sim}{x},t) + \int_0^t \psi(t-\tau)\underset{\sim}{B}(\underset{\sim}{x},\tau)d\tau$$

where $\underset{\sim}{E}$, $\underset{\sim}{H}$, $\underset{\sim}{B}$ are the usual electromagnetic field variables, $(\underset{\sim}{x},t) \in \Omega \times [0,T)$, $\Omega \subseteq R^3$ is a bounded region with smooth boundary $\partial\Omega$, a_o and b_o are positive constants, and ϕ, ψ are non-negative monotonically decreasing functions of t.

1. Introduction

In recent work [1] - [4] this author has derived stability and growth estimates for specific classes of solutions to initial-value problems associated with abstract integrodifferential equations of the form

$$\underset{\sim}{u}_{tt} - \underset{\sim}{N}\underset{\sim}{u} + \int_{-\infty}^{t} \underset{\sim}{K}(t-\tau)\underset{\sim}{u}(\tau)d\tau = \underset{\sim}{0}, \quad 0 \le t < T, \tag{1.1}$$

where $T > 0$ is an arbitrary real number; in this equation $\underset{\sim}{u} \in C^2([0,T); H_+)$ with $\underset{\sim}{u}_t \in C^1([0,T); H_+)$, and $\underset{\sim}{u}_{tt} \in C^0([0,T);H_-)$, where H_+, H_- are Hilbert spaces which are defined as follows: Let H be any real Hilbert space with inner-product $<,>$ and let $H_+ \subseteq H$ (algebraically and topologically) with H_+ dense in H; denote the inner-product on H_+ by $<,>_+$. Then H_- is the completion of H under the norm

$$||\underset{\sim}{w}||_- = \sup_{\underset{\sim}{v}\in H_+} \frac{|<\underset{\sim}{v},\underset{\sim}{w}>|}{||\underset{\sim}{v}||_+} \tag{1.2}$$

If we let $L(H_+,H_-)$ denote the space of bounded linear operators from H_+ into H_- then in (1.1) we only require that

(i) $\underset{\sim}{N} \in L(H_+,H_-)$ is symmetric and

(ii) $\underset{\sim}{K}(t)$, $\underset{\sim}{K}_t(t) \in L^2((-\infty,\infty); L(H_+,H_-))$

where $\underset{\sim}{K}_t$ denotes the strong operator derivative of $\underset{\sim}{K}$; no definiteness assumptions are placed on $\underset{\sim}{N}$ and thus the initial-value problem obtained by appending to (1.1) the initial data

$$\underset{\sim}{u}(0) = \underset{\sim}{f},\ \underset{\sim}{u}_t(0) = \underset{\sim}{g};\ \underset{\sim}{f},\underset{\sim}{g} \in H_+ \tag{1.3a}$$

and the prescription of the past history, i.e.,

$$\underset{\sim}{u}(\tau) = \underset{\sim}{U}(\tau),\ -\infty < \tau < 0 \tag{1.3b}$$

is, in general, non well-posed. If, however, we restrict our attention to classes of bounded solutions to (1.1) - (1.3) of the form

$$\mathcal{N} = \{\underset{\sim}{v} \in C^2([0,T);\ H_+) \mid \sup_{[0,T)} \|\underset{\sim}{v}(t)\|_+ \le N^2\}$$

for some arbitrary real number N, then it is possible to derive both stability and growth estimates for solutions $\underset{\sim}{u} \in \mathcal{N}$ under the assumption that $\underset{\sim}{K}(0)$ satisfies

$$-\langle\underset{\sim}{v},\underset{\sim}{K}(0)\underset{\sim}{v}\rangle \ge \kappa\|\underset{\sim}{v}\|_+^2,\ \forall v \in H_+ \tag{1.4a}$$

with

$$\kappa \ge \gamma T \sup_{[0,T)} \|\underset{\sim}{K}_t(t)\|_{\mathcal{L}(H_+,H_-)} \tag{1.4b}$$

In (1.4b) γ represents the embedding constant, i.e., as we assume that $H_+ \subseteq H$ topologically, $\|\underset{\sim}{v}\| \le \gamma\|\underset{\sim}{v}\|_+$ for some $\gamma > 0$ and all $\underset{\sim}{v} \in H_+$.

The technique used in [1] - [3] to derive the aforementoned stability and growth estimates for solutions $\underset{\sim}{u} \in \mathcal{N}$ of the abstract system (1.1), (1.3a), (1.3b) is based on a logarithmic convexity argument first employed by Knops and Payne [5] for

the abstract wave equation obtained from (1.1) by setting $\underset{\sim}{K}(t) \equiv 0$; a different logarithmic convexity argument was employed by this author in [4] to derive continuous data dependence theorems for the system (1.1), (1.3a), (1.3b). The results obtained in [2] - [4] are applied in those papers to obtain growth, stability, and continuous data dependence theorems for solutions to initial-value problems associated with the equations of motion for linear isothermal viscoelastic materials; the spaces H, H_+, and H_-, as well as the operators $\underset{\sim}{N}$ and $\underset{\sim}{K}(t)$, are constructed and no definiteness assumptions are made on the initial value of the relaxation tensor. In the case of a one-dimensional homogeneous (isothermal) linear viscoelastic body, it is shown in [3] that the conditions (1.4a), (1.4b) are equivalent to the requirement that

$$g'(0) \le -\kappa \text{ with } \kappa > \gamma T \sup_{[0,T)} |\ddot{g}(t)| \tag{1.5}$$

where $g(t)$ is the relaxation function of the material.

More recently we have turned our attention to the way in which integrodifferential equations arise in the theory of polarized non-conducting material dielectrics, i.e., in [6] we have considered the following problem: Let $\underset{\sim}{E}$, $\underset{\sim}{B}$, $\underset{\sim}{P}$, and $\underset{\sim}{D}$ denote, respectively, the electric field vector, the magnetic flux density, the polarization vector, and the electric displacement in a non-conducting medium; the polarization and electric displacement vectors are related via

$$\underset{\sim}{D} = \varepsilon_o \underset{\sim}{E} + \underset{\sim}{P}, \ \varepsilon_o \equiv \text{const.} \tag{1.6}$$

If (x^i,t), $i = 1,2,3$, denotes a Lorentz reference frame, with the (x^i) rectangular Cartesian coordinates and t the time parameter, then Maxwell's equations have the local form

$$\frac{\partial \underset{\sim}{B}}{\partial t} + \text{curl } \underset{\sim}{E} = 0, \ \text{div } \underset{\sim}{B} = 0 \tag{1.7a}$$

$$\text{curl } \underset{\sim}{H} - \frac{\partial \underset{\sim}{D}}{\partial t} = 0, \ \text{div } \underset{\sim}{D} = 0 \tag{1.7b}$$

whenever the density of free current $\underset{\sim}{J}_F = \underset{\sim}{0}$, the magnetization $\underset{\sim}{M} = \underset{\sim}{0}$, and the density of free charge $Q_F = 0$; in (1.7b), $\underset{\sim}{H}$ represents the magnetic intensity and is related to the magnetic flux density via

$$\underset{\sim}{H} = \mu_o^{-1} \underset{\sim}{B} \tag{1.8}$$

where $\varepsilon_o \mu_o = c^{-2}$, c being the speed of light in a vacuum. A determinate system of equations for the fields appearing in Maxwell's equations is obtained by specifying a set of constitutive relations. For example, in a vacuum $\underset{\sim}{P} = \underset{\sim}{0}$ and the classical relations

$$\underset{\sim}{D} = \varepsilon_o \underset{\sim}{E}, \ \underset{\sim}{H} = \mu_o^{-1} \underset{\sim}{B} \tag{1.9}$$

apply, while in a rigid, linear, stationary nonconducting dielectric

$$\underset{\sim}{D} = \underset{\sim}{\varepsilon} \cdot \underset{\sim}{E}, \ \underset{\sim}{B} = \underset{\sim}{\mu} \cdot \underset{\sim}{H} \tag{1.10}$$

where $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{\mu}$ are constant second order tensors; the constitutive equations (1.10) were given by Maxwell in 1873 [7]. In [6] we considered the set of equations which define the dielectric as being a Maxwell-Hopkinson material, i.e., (1.10_2) and

$$\underset{\sim}{D}(t) = \varepsilon\underset{\sim}{E}(t) + \int_{-\infty}^{t}\phi(t-\tau)\underset{\sim}{E}(\tau)d\tau \tag{1.11}$$

where $\varepsilon > 0$ and $\phi(t)$ is a continuous monotonically decreasing function for $t \geq 0$; following a suggestion of Maxwell, Hopkinson [8] employed the constitutive equations (1.10_2), (1.11) in connection with his studies on the residual charge of the Leyden jar. It was demonstrated in [6] that (1.11), in conjunction with the local Maxwell equations (1.7a), (1.7b), implies that the following integrodifferential equations govern the evolution of the electric field and the electric displacement field, respectively, in a non-conducting material dielectric of Maxwell-Hopkinson type:

$$(\varepsilon\underset{\sim}{E} + \phi*\underset{\sim}{E})_{tt} = \mu^{-1}\underset{\sim}{\Delta}\underset{\sim}{E} \tag{1.12a}$$

and

$$\varepsilon\mu\underset{\sim}{D}_{tt} = \Delta\underset{\sim}{D} + \Phi*\Delta\underset{\sim}{D} \tag{1.12b}$$

where for any vector field $\underset{\sim}{V}$

$$(\phi*\underset{\sim}{V})_i(\underset{\sim}{x},t) = \int_0^t\phi(t-\tau)V_i(\underset{\sim}{x},\tau)d\tau \tag{1.13a}$$

and

$$\Delta\underset{\sim}{V} = \text{grad}\,(\text{div}\,\underset{\sim}{V}) - \text{curl curl}\,\underset{\sim}{V} \tag{1.13b}$$

The function $\Phi(t)$ in (1.12b) is given in terms of the specified memory function $\phi(t)$ by

$$\Phi(t) = \sum_{n=1}^{\infty} (-1)^n \phi^n(t) \tag{1.14a}$$

where the $\phi^n(t)$, $n \geq 1$, are determined by the recursion relations

$$\phi^1(t) = \varepsilon^{-1}\phi(t) \tag{1.14b}$$

$$\phi^n(t) = \int_0^t \phi^1(t-\tau)\phi^{n-1}(\tau)d\tau, \; n \geq 2$$

Together with (1.12a), (1.12b) we considered initial data of the form

$$\underset{\sim}{E}(\underset{\sim}{x},0) = \underset{\sim}{E}_o(x), \quad \underset{\sim}{E}_t(\underset{\sim}{x},0) = \underset{\sim}{E}_1(\underset{\sim}{x}) \tag{1.15a}$$

$$\underset{\sim}{D}(\underset{\sim}{x},0) = \underset{\sim}{D}_o(x), \quad \underset{\sim}{D}_t(\underset{\sim}{x},0) = \underset{\sim}{D}_1(\underset{\sim}{x}) \tag{1.15b}$$

for $\underset{\sim}{x} \in \Omega$ (a bounded region in R^3 with smooth boundary $\partial\Omega$) and homogeneous boundary data of the form

$$\underset{\sim}{E}(\underset{\sim}{x},t) = \underset{\sim}{D}(\underset{\sim}{x},t) = \underset{\sim}{0}, \quad (\underset{\sim}{x},t) \in \partial\Omega \times (-\infty,T) \tag{1.16}$$

The functions $\underset{\sim}{E}_o, \ldots, \underset{\sim}{D}_1$ were taken to be continuous on $\bar{\Omega}$.

By introducing suitable Hilbert spaces H, H_+, H_- and operators $N \in \mathcal{L}(H_+,H_-)$ and $\underset{\sim}{K}(t) \in L^2((-\infty,\infty); \mathcal{L}(H_+,H_-))$ we were able in [6] to treat the initial-boundary value problem for $\underset{\sim}{D}$, i.e. (1.12b), (1.15b), (1.16_2), as a special case of the abstract initial-value problem (1.1), (1.2) (in [6] we assumed

that $\underset{\sim}{D}(\tau) = \underset{\sim}{0}, -\infty < \tau < 0)$. From the stability and growth estimates derived for the electric displacement field $\underset{\sim}{D}$, corresponding estimates were then derived for the electric field $\underset{\sim}{E}$ by employing the relation

$$\underset{\sim}{E}(t) = \varepsilon^{-1}\underset{\sim}{D}(t) + \varepsilon^{-1}\int_0^t \Phi(t-\tau)\underset{\sim}{D}(\tau)d\tau \tag{1.17}$$

which is obtaining by inverting the Maxwell-Hopkinson relation (1.11) via the usual technique of successive approximations.

The constitutive relations associated with the Maxwell-Hopkinson theory, i.e., (1.10_2) and (1.11), embody three basic simplifying assumptions: they are linear, they effect an a priori separation of electric and magnetic effects, and they do not allow for magnetic memory effects. As early as 1912 Volterra [9] proposed extending the Maxwell-Hopkinson theory to treat the case where the dielectric is anisotropic, non-linear, and magnetized; his constitutive relations were of the form

$$\underset{\sim}{D}(\underset{\sim}{x},t) = \underset{\sim}{\varepsilon}\cdot\underset{\sim}{E}(\underset{\sim}{x},t) + \underset{\underset{\sim}{-\infty}}{\overset{t}{\mathcal{D}}}(\underset{\sim}{E}(\underset{\sim}{x},\tau)) \tag{1.18a}$$

$$\underset{\sim}{B}(\underset{\sim}{x},t) = \underset{\sim}{\mu}\cdot\underset{\sim}{H}(\underset{\sim}{x},t) + \underset{-\infty}{\overset{t}{\mathcal{B}}}(\underset{\sim}{H}(\underset{\sim}{x},\tau) \tag{1.18b}$$

and it can be shown that (1.18a) reduces to (1.11) if the functional $\underset{\sim}{\mathcal{D}}$ is linear and isotropic and the body satisfies various restrictions which follow from considerations of material symmetry. Of course, (1.18a), (1.18b) still effect an a priori separation of electric and magnetic effects and,

as pointed out by Toupin and Rivlin [10], such a separation is inadequate with respect to predicting such phenomena as the Faraday effect in dielectrics. Thus, Toupin and Rivlin postulate [10] constitutive equations of the form

$$\underset{\sim}{D}(t) = \sum_{\nu=0}^{n} \underset{\sim}{a}_{\nu} \cdot \underset{\sim}{E}^{(\nu)}(t) + \sum_{\nu=0}^{n} \underset{\sim}{c}_{\nu} \cdot \underset{\sim}{B}^{(\nu)}(t) \tag{1.19a}$$

$$+ \int_{-\infty}^{t} \underset{\sim}{\phi}_1(t,\tau)\cdot\underset{\sim}{E}(\tau)d\tau + \int_{-\infty}^{t} \phi_2(t,\tau) \cdot \underset{\sim}{B}(\tau)d\tau$$

$$\underset{\sim}{H}(t) = \sum_{\nu=0}^{n} \underset{\sim}{d}_{\nu}\cdot E^{(\nu)}(t) + \sum_{\nu=0}^{n} \underset{\sim}{b}_{\nu} \cdot \underset{\sim}{B}^{(\nu)}(t)$$

$$+ \int_{-\infty}^{t} \underset{\sim}{\psi}_1(t,\tau) \cdot \underset{\sim}{B}(\tau)d\tau + \int_{-\infty}^{t} \underset{\sim}{\psi}_2(t,\tau) \cdot \underset{\sim}{E}(\tau)d\tau$$

where $\underset{\sim}{E}^{(\nu)}(t) = d^{\nu}\underset{\sim}{E}(t)/dt^{\nu}$ and $\underset{\sim}{a}_{\nu},\ldots, \underset{\sim}{d}_{\nu}$ are constant tensors; the kernels $\underset{\sim}{\phi}_1,\ldots, \underset{\sim}{\psi}_2$ are taken to be continuous tensor functions of t and τ which satisfy growth conditions of the form

$$\phi_1(t,\tau) < \underset{\sim}{c}/(t-\tau)^{1+\rho}, \ \rho > 0 \tag{1.20}$$

Toupin and Rivlin [10] also assume that the dielectric does not exhibit aging and as a consequence it follows that $\underset{\sim}{D}(t)$ and $\underset{\sim}{H}(t)$ are periodic functions whenever $\underset{\sim}{E}(t)$ and $\underset{\sim}{B}(t)$ are; this latter result, when combined with the hypothesized growth estimates on the kernel functions, e.g. (1.20), and early results of Volterra on the theory of functionals [9], yields the conclusion that $\underset{\sim}{\phi}_1,\ldots, \underset{\sim}{\psi}_2$ depend on t and τ only through

the difference $t-\tau$ (the converse of this result is also true). Toupin and Rivlin [10] then prove that if the dielectric exhibits holohedral isotropy, i.e., if it admits as its group of material symmetry transformations the full orthogonal group, then $\underset{\sim}{E}(t)$ may be eliminated from (1.19b) and $\underset{\sim}{B}(t)$ may be eliminated from (1.19a); for a holohedral isotropic dielectric the constitutive equations (1.19a), (1.19b) are, therefore, reducible to the uncoupled set

$$\underset{\sim}{D}(t) = \sum_{\nu=0}^{n} a_\nu \underset{\sim}{E}^{(\nu)}(t) + \int_{-\infty}^{t} \phi(t-\tau)\underset{\sim}{E}(\tau)d\tau \tag{1.21a}$$

$$\underset{\sim}{H}(t) = \sum_{\nu=0}^{n} b_\nu \underset{\sim}{B}^{(\nu)}(t) + \int_{-\infty}^{t} \psi(t-\tau)\underset{\sim}{B}(\tau)d\tau \tag{1.21b}$$

where $\phi = \phi_1$, $\psi = \psi_1$ and where (due to the assumption of holohederal isotropy) $\underset{\sim}{a}_\nu$, $\underset{\sim}{b}_\nu$, $\underset{\sim}{\phi}_1$ and $\underset{\sim}{\psi}_1$ are all proportional to the identity tensor and thus appear as scalars in (1.21a), (1.21b).

In this paper we will examine the special case of (1.21a), (1.21b) which corresponds to the assumptions

(H_1) $a_\nu = 0$, $b_\nu = 0$, $\nu \geq 1$

(H_2) $\underset{\sim}{E}(\tau) \equiv 0$, $\underset{\sim}{B}(\tau) \equiv 0$, $-\infty < \tau < 0$,

namely,

$$\underset{\sim}{D}(t) = a_o\underset{\sim}{E}(t) + \int_0^t \phi(t-\tau)\underset{\sim}{E}(\tau)d\tau \tag{1.22a}$$

$$\underset{\sim}{H}(t) = b_o\underset{\sim}{B}(t) + \int_0^t \psi(t-\tau)\underset{\sim}{B}(\tau)d\tau \tag{1.22b}$$

This special case of a holohedral isotropic non-conducting material dielectric still embodies a separation of electric and magnetic effects in the constitutive theory but generalizes the Maxwell-Hopkinson theory in that magnetic memory effects are taken into account through the presence of the kernel function $\psi(t)$. In the next section we will formulate an initial-boundary value problem for the electric displacement field $\underset{\sim}{D}(t)$ in a holohedral isotropic dielectric; provided $\psi(0) \neq 0$, $\underset{\sim}{D}(t)$ will be shown to satisfy a (non-homogeneous) damped integrodifferential equation. By introducing suitable Hilbert spaces and operators, the initial-boundary value problem for $\underset{\sim}{D}(t)$ is easily demonstrated to be equivalent to an initial value problem for an abstract damped integro-differential equation and growth estimates for specific classes of solutions to this abstract problem are then obtained by employing a suitable logarithmic convexity argument. When $\underset{\sim}{D}(0) \neq \underset{\sim}{0}$ the growth estimates obtained depend on hypotheses concerning the relative magnitudes of certain measures of the size of the initial data (e.g., the initial energy) and the strong operator norm of the kernel of the relevant integral operator; in each case, however, the basic hypothesis employed is a coerciveness assumption (on the initial value of the kernel of the integral operator) of the type represented by (1.4a), (1.4b).

2. An Initial-Boundary Value Problem for Holohedral Isotropic Dielectrics

Let (x^i,t) be a fixed Lorentz reference frame; the local

forms of Maxwell's equations are then given by (1.7a), (1.7b). Let $\Omega \subseteq R^3$ be a bounded region with boundary $\partial\Omega$ and assume that $\partial\Omega$ is sufficiently smooth so that the divergence theorem may be applied. Finally, assume that Ω is filled with a holohedral isotropic non-conducting dielectric material which is non-deformable and which satisfies hypotheses H_1 and H_2 of §1; in Ω, therefore, the electromagnetic field satisfies constitutive relations of the form (1.22a), (1.22b) where we assume that $a_o > 0$, $b_o > 0$ and $\phi(t)$, $\psi(t)$ are monotonically decreasing functions which are (at least) twice continuously differentiable on $[0,\infty)$ with $\psi^{(3)}(t)$ a bounded integrable function on $[0,\infty)$. The basic result of this section is the following:

<u>Theorem II.1</u> The evolution of the electric displacement field $\underset{\sim}{D}(\underset{\sim}{x},t)$ in any holohedral isotropic non-conducting material dielectric (which conforms to the constitutive hypotheses (1.22a), (1.22b)) is governed by the system of damped integrodifferential equations

$$\frac{\partial^2 D_i}{\partial t^2} + \psi(0)\frac{\partial D_i}{\partial t} - b_o\dot{\psi}(0)[c_o\delta_{ij}\delta_{j\ell}\frac{\partial^2 D_k}{\partial x_j\partial x_\ell} - D_i] \qquad (2.1)$$

$$+ b_o\int_0^t(\ddot{\psi}(t-\tau)D_i(\tau) - \Phi_o(t-\tau)\delta_{ik}\delta_{j\ell}\frac{\partial^2 D_k(\tau)}{\partial x_j\partial x_\ell})d\tau$$

$$= b_o\dot{\psi}(t)D_i(0),\ i = 1,2,3$$

where $c_o = 1/a_o\dot{\psi}(0)$ and $\Phi_o(t) = \Phi(t)/a_o$.

Remark In (2.1) $\Phi(t)$ is given in terms of the memory function $\phi(t)$ by (1.14a) and (1.14b) - with ε replaced by a_o - and $\Psi(t)$ is defined in terms of $\psi(t)$ in an analogous manner, i.e., by (1.14a) and (1.14b) with $\phi(t)$ replaced by $\psi(t)$ and ε replaced by b_o.

Proof By using successive approximations we may invert the constitutive relations (1.22a) and (1.22b) to obtain, respectively,

$$\underset{\sim}{E}(t) = \frac{1}{a_o}\underset{\sim}{D}(t) + \frac{1}{a_o}\int_0^t \Phi(t-\tau)\underset{\sim}{D}(\tau)d\tau \tag{2.2a}$$

$$\underset{\sim}{B}(t) = \frac{1}{b_o}\underset{\sim}{H}(t) + \frac{1}{b_o}\int_0^t \Psi(t-\tau)\underset{\sim}{H}(\tau)d\tau \tag{2.2b}$$

with $\Phi(t)$ and $\Psi(t)$ defined in terms of $\phi(t)$ and $\psi(t)$, respectively, as indicated in the above remark. Now, from (2.2a) and the second Maxwell relation in (1.7b) we have div $\underset{\sim}{E}(t) = 0$ and, therefore,

$$\underset{\sim\sim}{\Delta E}(t) = -\text{ curl curl }\underset{\sim}{E}(t) \tag{2.3}$$

From (2.2b), however, and the first Maxwell relation in (1.7a)

$$\text{curl }\underset{\sim}{E}(t) = -\underset{\sim}{B}_t = -\frac{1}{b_o}H_t - \frac{1}{b_o}\Psi(0)\underset{\sim}{H}(t) \tag{2.4}$$
$$-\int_0^t \Psi_t(t-\tau)\underset{\sim}{H}(\tau)d\tau$$

Therefore,

$$-\text{ curl curl }\underset{\sim}{E}(t) = \frac{1}{b_o}(\text{curl } H)_t + \frac{1}{b_o}\Psi(0)(\text{curl }\underset{\sim}{H}(t)) \tag{2.5}$$
$$+\int_0^t \Psi_t(t-\tau)\text{curl }\underset{\sim}{H}(\tau)d\tau$$

$$= \frac{1}{b_o} \underset{\sim}{D}_{tt} + \frac{1}{b_o} \Psi(0)\underset{\sim}{D}_t$$

$$+ \int_0^t \Psi_t(t-\tau)\underset{\sim}{D}_\tau(\tau)d\tau$$

where the second relation in (2.5) follows from the first Maxwell equation in (1.7b). Combining (2.5_2) with (2.3) and then employing (2.2a) we obtain

$$\underset{\sim}{D}_{tt} + \Psi(0)\underset{\sim}{D}_t + b_o \int_0^t \Psi_t(t-\tau)\underset{\sim}{D}_\tau(\tau)d\tau \tag{2.6}$$

$$= \frac{b_o}{a_o} \underset{\sim}{\Delta}\, \underset{\sim}{D}(t) + \frac{b_o}{a_o} \int_0^t \Phi(t-\tau)\underset{\sim}{\Delta}\underset{\sim}{D}(\tau)d\tau$$

However,

$$\int_0^t \Psi_t(t-\tau)\underset{\sim}{D}_\tau(\tau)d\tau = - \int_0^t \Psi_\tau(t-\tau)\underset{\sim}{D}_\tau(\tau)d\tau \tag{2.7}$$

$$= - \Psi_\tau(t-\tau)\underset{\sim}{D}(\tau)\Big|_0^t$$

$$+ \int_0^t \Psi_{\tau\tau}(t-\tau)\underset{\sim}{D}(\tau)d\tau$$

$$= \dot{\Psi}(0)\underset{\sim}{D}(t) - \dot{\Psi}(t)\underset{\sim}{D}(0)$$

$$+ \int_0^t \Psi_{\tau\tau}(t-\tau)\underset{\sim}{D}(\tau)d\tau$$

Substituting (2.7_3) into (2.6) and using the fact that

$\Psi_{\tau\tau}(t-\tau) = \Psi_{tt}(t-\tau)$ we easily obtain

$$\underset{\sim}{D}_{tt} + \Psi(0)\underset{\sim}{D}_t + b_o\dot{\Psi}(0)(\underset{\sim}{I} - c_o\,\underset{\sim}{\Delta})\underset{\sim}{D}(t) \tag{2.8}$$

$$+ b_o \int_0^t (\Psi_{tt}(t-\tau)\underset{\sim}{I} - \Phi_o(t-\tau)\underset{\sim}{\Delta})\underset{\sim}{D}(\tau)d\tau$$

$$= b_o\,\dot{\Psi}(t)\underset{\sim}{D}(0), \text{ on } \Omega \times [0,\infty),$$

where $c_o = 1/a_o\,\dot{\Psi}(0)$ and $\Phi_o(t) = \Phi(t)/a_o$; this establishes the required result.

Remark We are assuming that $\dot{\Psi}(0) \neq 0$ so that c_o is defined; if $\dot{\Psi}(0) = 0$ then the third expression on the left hand side of equation (2.1) reduces to

$$\frac{b_o}{a_o}\,\delta_{ij}\,\delta_{k\ell}\,\frac{\partial^2 D_k}{\partial x_j \partial x_\ell}$$

In conjunction with the integrodifferential equation (2.8) we consider initial and boundary data of the form

$$\underset{\sim}{D}(\underset{\sim}{x}, 0) = \underset{\sim}{D}_0(\underset{\sim}{x}),\ \underset{\sim}{D}_t(\underset{\sim}{x},0) = \underset{\sim}{D}_1(\underset{\sim}{x}),\ \underset{\sim}{x} \in \bar{\Omega} \tag{2.9}$$

and

$$\underset{\sim}{D}(\underset{\sim}{x},t) = \underset{\sim}{0},\ (\underset{\sim}{x},t) \in \partial\Omega \times [0,\infty) \tag{2.10}$$

where $\underset{\sim}{D}_0$, $\underset{\sim}{D}_1$ are continuous on $\bar{\Omega}$. At this point it is convenient to recast the initial-boundary value problem (2.8),

(2.9), (2.10) as an initial value problem for an integrodifferential equation in Hilbert space. As in [6] we let $C_o^\infty(\Omega)$ denote the set of three dimensional vector fields with compact support in Ω whose components are in $C^\infty(\Omega)$. We define the Hilbert space H to be the completion of $C_o^\infty(\Omega)$ under the norm induced by the inner product

$$\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle_H \equiv \int_\Omega v_i w_i \, d\underset{\sim}{x} \tag{2.11}$$

while the Hilbert space H_+ is taken to be the completion of $C_o^\infty(\Omega)$ under the norm induced by the inner product

$$\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle_{H_+} \equiv \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, d\underset{\sim}{x} \tag{2.12}$$

Finally, H_- is the Hilbert space obtained by completing $C_o^\infty(\Omega)$ under the norm

$$||\underset{\sim}{v}||_{H_-} \equiv \sup_{\underset{\sim}{w} \in H_+} \left[\left| \int_\Omega v_i w_i \, d\underset{\sim}{x} \right| / \left(\int_\Omega \frac{\partial w_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} \right)^{\frac{1}{2}} d\underset{\sim}{x}) \right] \tag{2.13}$$

It may be established that $H_+ \subseteq H$ (both topologically and algebraically) and that H_+ is dense in H; the spaces H, H_+, H_- which are defined above are commonly denoted by $L_2(\Omega)$, $H_o^1(\Omega)$, and $H^{-1}(\Omega)$, respectively. We denote by ω the embedding constant for the inclusion map $i: H_+ \to H$.

Operators $\hat{\underset{\sim}{L}} \in L(H_+, H_-)$ and $\hat{\underset{\sim}{M}}(t) \in L^2((-\infty,\infty); L(H_+,H_-))$ are now defined as follows:

$$(\hat{\underset{\sim}{L}}\underset{\sim}{v})_i \equiv b_o\dot{\Psi}(0)[c_o\delta_{ik}\delta_{j\ell}\frac{\partial^2 v_k}{\partial x_j \partial x_\ell} - \delta_{ij}v_j], \ \underset{\sim}{v} \in H_+ \tag{2.14a}$$

$$(\hat{\underset{\sim}{M}}(t)\underset{\sim}{v})_i \equiv b_o[\ddot{\Psi}(t)\delta_{ij}v_j - \Phi_o(t)\delta_{ik}\delta_{j\ell}\frac{\partial^2 v_k}{\partial x_j \partial x_\ell}] \begin{cases} \underset{\sim}{v} \in H_+ \\ t \in (-\infty,\infty) \end{cases} \tag{2.14b}$$

It follows directly from the definitions (2.11) - (2.13), (2.14a), (2.14b) and the smoothness assumptions on the memory functions $\phi(t)$, $\psi(t)$ that

(i) $\hat{\underset{\sim}{L}} \in L_S(H_+, H_-)$, $\hat{\underset{\sim}{M}}(t) \in L_S(H_+, H_-)$, $t \in (-\infty,\infty)$

(ii) $\hat{\underset{\sim}{M}}_t(\cdot) \in L^2((-\infty,\infty); L(H_+, H_-))$

where $L_S(H_+, H_-)$ denotes the space of all symmetric bounded linear operators from H_+ into H_- and $\hat{\underset{\sim}{M}}_t$ is the strong operator derivative of $\hat{\underset{\sim}{M}}(\cdot)$. Using the definitions of H, H_+, H_- and the operators $\hat{\underset{\sim}{L}}$, $\hat{\underset{\sim}{M}}(t)$ given above we may rewrite the system (2.1), (2.9), (2.10) in the form

$$\underset{\sim}{D}_{tt} + \Psi(0)\underset{\sim}{D}_t - \hat{\underset{\sim}{L}}\underset{\sim}{D} + \int_0^t \hat{\underset{\sim}{M}}(t-\tau)\underset{\sim}{D}(\tau)d\tau = b_o\dot{\Psi}(t)\underset{\sim}{D}_o \tag{2.15}$$

$$\underset{\sim}{D}(0) = \underset{\sim}{D}_o, \ \underset{\sim}{D}_t(0) = \underset{\sim}{D}_1 \tag{2.16}$$

where $\underset{\sim}{D}_0$, $\underset{\sim}{D}_1 \in H_+$ and $\underset{\sim}{D} \in C^2([0,\infty); H_+)$ with $\underset{\sim}{D}_t \in C^1([0,\infty); H_+)$ and $\underset{\sim}{D}_{tt} \in C([0,\infty); H_-)$. Actually, we shall be interested in solutions of (2.15), (2.16) on finite time intervals of the form $[0,T)$ where T, $0 < T < \infty$, is an arbitrary real number;

this suggests that we examine the following abstract initial value problem: Let H, H_+ be Hilbert spaces with inner products $\langle , \rangle$ and $\langle , \rangle_+$, respectively, and assume that $H_+ \subseteq H$ (algebraically and topologically) with H_+ dense in H; define H_- as in (1.2). We consider solutions $\underset{\sim}{u} \in C^2([0,T); H_+)$ of the system

$$\underset{\sim}{u}_{tt} - \alpha \underset{\sim}{u}_t - \underset{\sim}{L}\underset{\sim}{u} + \int_0^t \underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)d\tau \tag{2.17}$$

$$= \beta(t)\underset{\sim}{u}_o, \quad 0 \le t < T$$

$$\underset{\sim}{u}(0) = \underset{\sim}{u}_o, \ \underset{\sim}{u}_t(0) = \underset{\sim}{u}_1 \ (\underset{\sim}{u}_o, \underset{\sim}{u}_1 \in H_+) \tag{2.18}$$

where $\alpha \neq 0$ is an arbitrary real constant, $\beta(t)$ is any real-valued function such that $\dot{\beta}(t)$ exists a.e. on $[0,T)$ and

(i') $\underset{\sim}{L} \in L_S(H_+, H_-)$

(ii') $\underset{\sim}{M}(\cdot), M_t(\cdot) \in L^2([0,T); L_S(H_+, H_-))$

We assume that $\underset{\sim}{u}_t \in C^1([0,T); H_+)$ and $\underset{\sim}{u}_{tt} \in C([0,T); H_-)$.

In §3 we will derive some growth estimates for solutions $\underset{\sim}{u}(t)$ of the system (2.17), (2.18), which lie in a certain uniformly bounded subset of $C^2([0,T); H_+)$; our estimates will be obtained under various combinations of the following hypotheses: $\alpha \begin{Bmatrix} > 0 \\ < 0 \end{Bmatrix}$, $\underset{\sim}{u}_o \begin{Bmatrix} = \underset{\sim}{0} \\ \neq \underset{\sim}{0} \end{Bmatrix}$ and $\beta(t) \begin{Bmatrix} = 0, & 0 \le t < T \\ \neq 0, & \text{on } [0,T) \end{Bmatrix}$

In §4 we will apply our results to the system consisting of

(2.1), (2.9), and (2.10); at no point in this work do we make any definiteness assumptions on the operators $\underset{\sim}{L}$ or $\underset{\sim}{M}(t)$, $t \in [0,T)$.

3. Some Growth Estimates for a Damped Integrodifferential Equation

We begin with some preliminary definitions. Let $N > 0$ be an arbitrary positive real number and let

$$\mathcal{N} = \{\underset{\sim}{u} \in C^2([0,T); H_+) \mid \sup_{[0,T)} ||\underset{\sim}{u}||_+ \le N\}$$

Let $K(t) = \frac{1}{2}||\underset{\sim}{u}_t||^2$ denote the kinetic energy associated with solutions $\underset{\sim}{u}$ of the system (2.17), (2.18) and $P(t) = -\frac{1}{2}\langle\underset{\sim}{u}, \underset{\sim}{N}\underset{\sim}{u}\rangle$ the potential energy; then $E(t) \equiv K(t) + P(t)$ is the total energy. Finally, let γ and t_o be arbitrary non-negative real numbers and define

$$F(t;\gamma,t_o) \equiv ||\underset{\sim}{u}(t)||^2 + \gamma(t+t_o)^2, \quad 0 \le t < T \tag{3.1}$$

The various growth estimates we derive in this section all follow from the following basic

<u>Lemma</u> Let $\underset{\sim}{u} \in \mathcal{N}$ be any solution of (2.17), (2.18) and suppose that $\underset{\sim}{M}(0)$ satisfies

$$-\langle\underset{\sim}{v}, \underset{\sim}{M}(0)\underset{\sim}{v}\rangle \ge \kappa||\underset{\sim}{v}||_+^2, \quad \forall \underset{\sim}{v} \in H_+ \tag{3.2a}$$

with

$$\kappa \geq \gamma T \sup_{[0,T)} ||\underset{\sim}{M}_t||_{L(H_+,H_-)} \tag{3.2b}$$

Then there exists $\mu > 0$ such that for all t, $0 \leq t < T$

$$\begin{aligned} FF'' - F'^2 \geq & -2F(2E(0) + \mu) + \alpha FF' \\ & - 2\alpha F(\gamma(t+t_o) + 4\int_0^t K(\tau)d\tau) \\ & + 2F(2\int_0^t \dot{\beta}(\tau)<\underset{\sim}{u},\underset{\sim}{u}_o>d\tau - \beta(t)<\underset{\sim}{u},\underset{\sim}{u}_o>) \\ & + 4F\beta(0)||\underset{\sim}{u}_o||^2 \end{aligned} \tag{3.3}$$

<u>Proof</u> From the definition of $F(t;\gamma,t_o)$, i.e. (3.1), we easily compute

$$F'(t;\gamma,t_o) = 2<\underset{\sim}{u},\underset{\sim}{u}_t> + \gamma(t+t_o) \tag{3.4}$$

and

$$\begin{aligned} F''(t;\gamma,t_o) = & 2||\underset{\sim}{u}_t||^2 + 2\alpha<\underset{\sim}{u},\underset{\sim}{u}_t> \\ & + 2<\underset{\sim}{u}, \underset{\sim}{L}\underset{\sim}{u}> - 2<\underset{\sim}{u}, \int_0^t \underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)d\tau \\ & + 2\beta(t)<\underset{\sim}{u},\underset{\sim}{u}_o> + 2\gamma, \end{aligned} \tag{3.5}$$

where we have made use of (2.17) in (3.5). Using the definitions of $K(t)$, $E(t)$, we may rewrite (3.5) in the form

$$\begin{aligned} F''(t;\gamma,t_o) = & 2\alpha<\underset{\sim}{u},\underset{\sim}{u}_t> + 2\beta(t)<\underset{\sim}{u},\underset{\sim}{u}_o> \\ & - 2<\underset{\sim}{u}, \int_0^t \underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)d\tau \\ & + 4(2K(t) + \gamma) - 2(2E(0) + \gamma) - 4(E(t) - E(0)) \end{aligned} \tag{3.6}$$

However, for any τ, $0 \le \tau \le t < T$

$$E'(\tau) = \langle \underset{\sim}{u}_\tau, \underset{\sim}{u}_{\tau\tau} \rangle - \langle \underset{\sim}{u}_\tau, \underset{\sim}{L}\underset{\sim}{u} \rangle \tag{3.7}$$

$$= \alpha \| \underset{\sim}{u}_\tau \|^2 + \beta(\tau) \langle \underset{\sim}{u}_\tau, \underset{\sim}{u}_o \rangle$$

$$- \langle \underset{\sim}{u}_\tau, \int_0^\tau \underset{\sim}{M}(\tau-\sigma) u(\sigma) d\sigma \rangle$$

where (3.7_2) follows by taking the inner product (in H) of (2.17) with $\underset{\sim}{u}_\tau$. Therefore,

$$E'(\tau) = 2\alpha K(\tau) + \beta(\tau) \langle \underset{\sim}{u}_\tau, \underset{\sim}{u}_o \rangle \tag{3.8}$$

$$- \frac{d}{d\tau} \langle \underset{\sim}{u}(\tau), \int_0^\tau \underset{\sim}{M}(\tau-\sigma) \underset{\sim}{u}(\sigma) d\sigma \rangle$$

$$+ \langle \underset{\sim}{u}(\tau), \int_0^\tau \underset{\sim}{M}_\tau(\tau-\sigma) \underset{\sim}{u}(\sigma) d\sigma \rangle$$

$$+ \langle \underset{\sim}{u}(\tau), \underset{\sim}{M}(0) \underset{\sim}{u}(\tau) \rangle$$

Integrating this last result from zero to t and substituting for $E(t) - E(0)$ in (3.6) we obtain

$$F''(t;\gamma,t_o) = 2\alpha \langle \underset{\sim}{u}, \underset{\sim}{u}_t \rangle + 2\beta(t) \langle \underset{\sim}{u}, \underset{\sim}{u}_o \rangle \tag{3.9}$$

$$+ 2 \langle \underset{\sim}{u}, \int_0^t \underset{\sim}{M}(t-\tau) \underset{\sim}{u}(\tau) d\tau \rangle$$

$$+ 4(2K(t) + \gamma) - 2(2E(0) + \gamma)$$

$$- 8\alpha \int_0^t K(\tau) d\tau - 4 \int_0^t \beta(\tau) \langle \underset{\sim}{u}_\tau, \underset{\sim}{u}_o \rangle d\tau$$

$$- 4 \int_0^t \langle \underset{\sim}{u}(\tau), \int_0^\tau \underset{\sim}{M}_\tau(\tau-\sigma) \underset{\sim}{u}(\sigma) d\sigma \rangle d\tau$$

$$- 4 \int_0^t \langle \underset{\sim}{u}(\tau), \underset{\sim}{M}(0) \underset{\sim}{u}(\tau) \rangle d\tau$$

Therefore,

$$FF'' = F'^2 = 4F(2K(t) + \gamma) - F'^2 - 2F(2E(0) + \gamma) \tag{3.10}$$

$$+ 2\alpha F(<\underset{\sim}{u},\underset{\sim}{u}_t> - 4\int_0^t K(\tau)d\tau)$$

$$+ 2F(\beta(t)<\underset{\sim}{u},\underset{\sim}{u}_o> - 2\int_0^t \beta(\tau)<\underset{\sim}{u}_\tau,\underset{\sim}{u}_o>d\tau)$$

$$+ 2F<\underset{\sim}{u},\int_0^t \underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)d\tau>$$

$$+ 4F\int_0^t <\underset{\sim}{u}(\tau), \int_0^\tau \underset{\sim}{M}_\tau(\tau-\sigma)\underset{\sim}{u}(\sigma)d\sigma>d\tau$$

$$- 4F\int_0^t <\underset{\sim}{u}(\tau), \underset{\sim}{M}(0)\underset{\sim}{u}(\tau)>d\tau$$

However, from (3.1), (3.4), the definition of $K(t)$, and the Schwarz inequality it follows that

$$G(t;\gamma,t_o) \equiv 4F(t;\gamma,t_o)(2K(t) + \gamma) - F'^2(t;\gamma,t_o) \geq 0 \tag{3.11}$$

and, therefore, (3.10) yields the inequality

$$FF'' - F'^2 \geq - 2F(2E(0) + \gamma) \tag{3.12}$$

$$+ \alpha F(\frac{d}{dt}||\underset{\sim}{u}||^2 - 8\int_0^t K(\tau)d\tau)$$

$$+ 2F(2\int_0^t \dot{\beta}(\tau)<\underset{\sim}{u},\underset{\sim}{u}_o>d\tau - \beta(t)<\underset{\sim}{u},\underset{\sim}{u}_o>)$$

$$+ 4F\beta(0)||\underset{\sim}{u}_o||^2$$

$$+ 2F <\underset{\sim}{u}, \int_0^t \underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)d\tau>$$

$$- 4F\int_0^t <\underset{\sim}{u}(\tau), \int_0^\tau \underset{\sim}{M}_\tau(\tau-\sigma)\underset{\sim}{u}(\sigma)d\sigma>d\tau$$

$$- 4F\int_0^t <\underset{\sim}{u}(\tau), \underset{\sim}{M}(0)\underset{\sim}{u}(\tau)>d\tau$$

If we make note of the fact that

$$\frac{d}{dt}||\underset{\sim}{u}||^2 = F'(t;\gamma,t_o) - 2\gamma(t+t_o)$$

then we can rewrite (3.12) in the form

$$\begin{aligned} FF'' - F'^2 \geq & - 2F(2E(0) + \gamma) + \alpha FF' \qquad (3.13) \\ & - 2\alpha F(\gamma(t+t_o) + 4\int_0^t K(\tau)d\tau) \\ & + 2F(2\int_0^t \dot{\beta}(\tau)<\underset{\sim}{u},\underset{\sim}{u}_o>d\tau - \beta(t)<\underset{\sim}{u},\underset{\sim}{u}_o>) + 4F\beta(0)||\underset{\sim}{u}_o|| \\ & + 2F<\underset{\sim}{u}, \int_0^t \underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)d\tau> \\ & - 4F\int_0^t <\underset{\sim}{u}(\tau), \int_0^\tau \underset{\sim}{M}_\tau(\tau-\sigma)\underset{\sim}{u}(\sigma)d\sigma>d\tau \\ & - 4F \int_0^t<\underset{\sim}{u}(\tau), \underset{\sim}{M}(0)\underset{\sim}{u}(\tau)>d\tau \end{aligned}$$

In order to complete the proof of the Lemma we now use the hypotheses (3.2a), (3.2b) and the fact that $\underset{\sim}{u} \in N$ to bound, from below, the sum of the last three terms in (3.13). First of all

$$\begin{aligned} & |<\underset{\sim}{u}, \int_0^t \underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)d\tau>| \\ & \qquad \leq ||\underset{\sim}{u}(t)||\int_0^t||\underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)||d\tau \\ & \qquad \leq \omega||\underset{\sim}{u}(t)||_+ \int_0^t(||\underset{\sim}{M}(t-\tau)||_{L(H_+,H_-)})||\underset{\sim}{u}(\tau)||_+d\tau \\ & \qquad \leq \omega T(\sup_{[0,T)} ||\underset{\sim}{u}||_+)^2 \sup_{[0,T)} ||\underset{\sim}{M}(t)||_{L(H_+,H_-)} \\ & \qquad \leq \omega N^2 T \sup_{[0,T)} ||\underset{\sim}{M}(t)||_{L(H_+,H_-)} \end{aligned}$$

and thus, as $F(t;\gamma,t_o) \geq 0$, $0 \leq t < T$,

$$2F\langle \underset{\sim}{u}, \int_0^t \underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)d\tau\rangle \tag{3.14b}$$

$$\geq -\ 2\omega N^2 T \sup_{[0,T)} \|\underset{\sim}{M}(t)\|_{L(H_+,H_-)} F(t;\gamma,t_o)$$

Also,

$$-\ 4F\int_0^t \langle \underset{\sim}{u}(\tau),\ \underset{\sim}{M}(0)\underset{\sim}{u}(\tau)\rangle d\tau \tag{3.15}$$

$$\geq 4\kappa F\int_0^t \|\underset{\sim}{u}(\tau)\|_+^2\ d\tau$$

$$4\omega T \sup_{[0,T)} \|\underset{\sim}{M}_t\|_{L(H_+,H_-)} F\int_0^t \|\underset{\sim}{u}(\tau)\|_+^2\ d\tau\ .$$

by virtue of (3.2a) and (3.2b). Finally,

$$\int_0^t \langle \underset{\sim}{u}(\tau),\ \int_0^\tau \underset{\sim}{M}_\tau(\tau-\sigma)\underset{\sim}{u}(\sigma)d\sigma\rangle d\tau \tag{3.16a}$$

$$\leq \int_0^t |\langle \underset{\sim}{u}(\tau),\ \int_0^\tau \underset{\sim}{M}_\tau(\tau-\sigma)\underset{\sim}{u}(\sigma)d\sigma\rangle| d\tau$$

$$\leq \int_0^t \|\underset{\sim}{u}(\tau)\|(\int_0^\tau(\|\underset{\sim}{M}_\tau(\tau-\sigma)\|_{L(H_+,H_-)})\|\underset{\sim}{u}(\sigma)\|_+ d\sigma)d\tau$$

$$\leq \omega \sup_{[0,T)} \|\underset{\sim}{M}_t\|_{L(H_+,H_-)}\int_0^t \|\underset{\sim}{u}(\tau)\|_+(\int_0^\tau \|\underset{\sim}{u}(\sigma)\|_+ d\sigma)d\tau$$

$$\leq \omega \sup_{[0,T)} \|\underset{\sim}{M}_t\|_{L(H_+,H_-)}(\int_0^t \|\underset{\sim}{u}(\tau)\|_+ d\tau)^2$$

$$\leq \omega T \sup_{[0,T)} \|\underset{\sim}{M}_t\|_{L(H_+,H_-)}\int_0^t \|\underset{\sim}{u}(\tau)\|_+^2\ d\tau$$

from which we easily deduce that

$$- 4F\int_0^t \langle \underset{\sim}{u}(\tau), \int_0^\tau \underset{\sim}{M}_\tau(\tau-\sigma)\underset{\sim}{u}(\sigma)d\sigma\rangle d\tau \tag{3.16b}$$

$$\geq - 4\omega T \sup_{[0,T)} ||\underset{\sim}{M}_t||_{L(H_+,H_-)} F\int_0^t ||\underset{\sim}{u}(\tau)||_+^2 \, d\tau$$

Combining (3.13) with the estimates (3.14b), (3.15_2) and (3.16b) we obtain the required result, i.e., the estimate (3.3) with

$$\mu \equiv \gamma + \omega N^2 T \sup_{[0,T)} ||\underset{\sim}{M}(t)||_{L(H_+,H_-)} \tag{3.17}$$

Q.E.D.

With the preceding Lemma as a starting point we now begin our study of the growth behavior of solutions to (2.17), (2.18) which lie in the class N; in each of the cases examined below we assume that $\underset{\sim}{M}(0)$ satisfies (3.2a) for some $\kappa > 0$ which satisfies (3.2b)

<u>Case I</u>: $\underset{\sim}{u}_o = 0$ and $\alpha < 0$

In this case $E(0) = \frac{1}{2}||\underset{\sim}{u}_1||^2$ and the second expression on the right-hand side of (3.3), being non-negative, may be dropped. Therefore,

$$FF'' - F'^2 \geq - 2F(||\underset{\sim}{u}_1||^2 + \mu) - |\alpha|FF' \tag{3.18}$$

for all t, $0 \leq t < T$, where μ is given by (3.17). However, for γ, t_o arbitrary nonnegative real numbers,

$$\lambda\gamma t_0^2 \le \lambda||\underset{\sim}{u}(t)||^2 + \lambda\gamma(t+t_o)^2 \equiv \lambda F(t;\gamma;t_o) \tag{3.19}$$

for any $\lambda \ge 0$. If, in particular, we choose

$$\lambda = \lambda(\gamma;t_o) \equiv \frac{2(||\underset{\sim}{u}_1||^2 + \mu)}{\gamma t_o^2} \tag{3.20}$$

then for all t, $0 \le t < T$, and all nonnegative real numbers γ, t_o

$$2(||\underset{\sim}{u}_1||^2 + \mu) \le \lambda(\gamma;t_o)F(t;\gamma,t_o) \tag{3.21}$$

and (3.18) may be replaced by the estimate

$$FF'' - F'^2 \ge -\lambda(\gamma;t_o)F^2 - |\alpha|FF' \tag{3.22}$$

The differential inequality (3.22) now forms the basis for the following growth estimate:

<u>Theorem III.1</u> Let $\underset{\sim}{u} \in N$ be any solution of (2.17), (2.18) with $\underset{\sim}{u}_o = 0$ and $\alpha < 0$. Assume that $\underset{\sim}{M}(0)$ satisfies (3.2a), (3.2b) and that $T > 1/|\alpha|$. Then there exist real constants $\bar{\lambda} > 0$ and $A > 0$ and a real-valued function $g(t) \ge 0$, $0 \le t < T$, such that

$$||\underset{\sim}{u}||^2 \le Ae^{\frac{\bar{\lambda}}{|\alpha|}g(t)}, \quad 0 \le t < T \tag{3.23}$$

Furthermore,

(i) $g(0) = 0, \quad \lim_{t \to T^-} g(t) = 0$

(ii) $g(t)$ is (strictly) monotonically $\left\{ \begin{matrix} \text{increasing} \\ \text{decreasing} \end{matrix} \right\}$ for

$$\left\{ \begin{matrix} t< \\ t> \end{matrix} \right\} \frac{1}{|\alpha|} \ell n \left[\frac{T|\alpha|}{1-e^{-|\alpha|T}} \right]$$

<u>Proof</u> From (3.22) and Jensen's inequality we obtain the estimate,

$$F(t;\gamma,t_o) \le e^{-\frac{\lambda}{|\alpha|} t} [F(t_1;\gamma,t_o)e^{\frac{\lambda}{|\alpha|} t_1}]^{\delta} [F(t_2;\gamma,t_o)e^{\frac{\lambda}{|\alpha|} t_2}]^{1-\delta} \qquad (3.24)$$

(valid for $0 \le t_1 < t \le t_2 < T$) where

$$\delta(t) = (e^{-|\alpha|t} - e^{-|\alpha|t_2})/(e^{-|\alpha|t_1} - e^{-|\alpha|t_2}) \qquad (3.25)$$

The interval $[t_1,t_2] \subseteq [0,T)$ is any closed interval such that $F(t;\gamma,t_o) > 0$, $t_1 \le t \le t_2$. However, it is a simple consequence of (3.24) and the definition of $F(t;\gamma,t_o)$ that $F(t;\gamma,t_o) \equiv 0$ on $[0,T)$ if $F(\bar{t};\gamma,t_o) = 0$ for any $\bar{t} \in [0,T)$. Thus, without loss of generality, we may assume that $F(t;\gamma,t_o) > 0$, $0 \le t < T$. Taking $t_1 = 0$, $t_2 = T$ in (3.14) we obtain

$$F(t;\gamma,t_o) \le e^{\frac{-\lambda}{|\alpha|} t} [\gamma t_o^2]^{\bar{\delta}} [F(T;\gamma,t_o)e^{\frac{\lambda}{|\alpha|} T}]^{1-\bar{\delta}} \qquad (3.26)$$

where

$$\bar{\delta}(t) = (e^{-|\alpha|t} - e^{-|\alpha|T})/(1 - e^{-|\alpha|T}) \qquad (3.27)$$

We now choose $\gamma = 1/t_o^2$ and then take the limit in (3.26) as $t_o \to +\infty$. Clearly, as

$$F(t;\, 1/t_o^2,\, t_o) = ||\underset{\sim}{u}(t)||^2 + (\frac{t}{t_o} + 1)^2$$

$$\lim_{t_o \to +\infty} F(t;1/t_o^2,t_o) = ||\underset{\sim}{u}(t)||^2 + 1 \tag{3.28a}$$

for all $t \in [0,T)$. Also

$$\lim_{t_o \to +\infty} F(T;1/t_o^2,t_o) = \lim_{t_o \to +\infty} (||\underset{\sim}{u}(T)||^2 + (\frac{T}{t_o} + 1)^2)) \tag{3.28b}$$

$$\leq \omega^2 N^2 + 1$$

as $\underset{\sim}{u} \in N$. Finally

$$\lim_{t_o \to +\infty} \lambda(1/t_o^2;t_o) = \lim_{t_o \to +\infty} 2(||\underset{\sim}{u}_1||^2 + 1/t_o^2 + \bar{\mu}) \tag{3.28c}$$

$$= 2(||\underset{\sim}{u}_1||^2 + \bar{\mu}) \equiv \bar{\lambda}$$

where $\bar{\mu} = \omega N^2 T \sup_{[0,T)} ||\underset{\sim}{M}(t)||_{L(H_+,H_-)}$. Thus, with $\gamma = 1/t_o^2$ and $t_o \to +\infty$ in (3.26), we obtain the estimate

$$||\underset{\sim}{u}(t)||^2 \leq Ae^{\frac{\bar{\lambda}}{|\alpha|} g(t)}, \quad 0 \leq t < T, \tag{3.29}$$

where

$$A = \sup_{[0,T)} (\omega^2 N^2 + 1)^{1-\bar{\delta}(t)} \tag{3.30}$$

and

$$g(t) \equiv T(1-\bar{\delta}(t)) - t \tag{3.31}$$

From the definition of $\bar{\delta}(t)$ it is clear that $g(0) = 0$ and $\lim_{t\to T^-} g(t) = 0$. Also

$$\bar{\delta}'(t) = \frac{-|\alpha|e^{-|\alpha|t}}{1-e^{-|\alpha|T}} < 0, \quad 0 \le t < T, \tag{3.32}$$

so

$$g'(t) = -T\bar{\delta}'(t) - 1 = T|\bar{\delta}'(t)| - 1 \tag{3.33}$$

Thus

$$g'(t) \begin{Bmatrix} < 0 \\ = 0 \\ > 0 \end{Bmatrix} \text{ if and only if } |\bar{\delta}'(t)| \begin{Bmatrix} <1/T \\ =1/T \\ >1/T \end{Bmatrix} \tag{3.34}$$

However,

$$|\bar{\delta}'(t)| = |\alpha|e^{-|\alpha|t}/(1-e^{-|\alpha|T})$$

so

$$|\bar{\delta}'(t)| < 1/T \iff |\alpha|e^{-|\alpha|t} < \frac{1}{T}(1-e^{-|\alpha|T}) \tag{3.35}$$

From (3.35) it follows directly that

$$|\bar{\delta}'(t)| < 1/T \iff t > \frac{1}{|\alpha|} \ln\left[\frac{|\alpha|T}{1-e^{-|\alpha|T}}\right] \tag{3.36}$$

provided

$$|\alpha|T > 1-e^{-|\alpha|T} \tag{3.37}$$

(and this last condition is certainly satisfied if $T > 1/|\alpha|$). This completes the proof of the theorem.

Q.E.D

Remark Let $M > 0$ be chosen so that

$$\omega^2 N^2 + 1 < Me^{\frac{-\bar{\lambda}}{|\alpha|} T} \tag{3.38}$$

If we again set $\gamma = 1/t_o^2$ in (3.26) and take the limit as $t_o \to +\infty$ then we obtain the estimate

$$||\underset{\sim}{u}(t)||^2 \le M^{1-\bar{\delta}} e^{\frac{-\bar{\lambda}}{|\alpha|} t}, \quad 0 \le t < T \tag{3.39}$$

which shows that $||\underset{\sim}{u}||^2$ is bounded above by an exponentially decreasing function of t for all $t \in [0,T)$.

In contrast to the result contained in the statement of Theorem III.1,and the above remark, we have the following theorem concerning lower bounds for solutions $\underset{\sim}{u} \in N$ of (2.17), (2.18):

Theorem III.2 Let $\underset{\sim}{u} \in N$ be any solution of (2.17), (2.18) with $\underset{\sim}{u}_o = 0$ and $\alpha < 0$ and assume that $\underset{\sim}{M}(0)$ satisfies (3.2a), (3.2b). If $|\alpha| < 1$ then there exists $T > 0$ such that $||\underset{\sim}{u}||^2$ is bounded below by a monotonically increasing exponential function of t, $0 \le t < T$.

Proof We begin by integrating the differential inequality (3.22) according to the "tangent property" of convex functions-assuming that $F(t;\gamma,t_o) > 0$, $0 \le t < T$, where $T > 0$ is an arbitrary real number by the "tangent property" for convex functions we refer to the fact that the graph of a convex function[(1)] on $[0,T)$ lies above the tangent line to the curve at any point $\bar{t} \in [0,T)$. Thus, we obtain directly from (3.22) the estimate

(1) The inequality (3.22) and the assumption that $F(t;\gamma,t_o) > 0$ on $[0,T)$ imply that $\ell n(F(\sigma;\gamma,t_o)e^{-\lambda/\alpha^2})$ is a convex function of $\sigma = e^{-|\alpha|t}$ on $[0,T)$.

$$F(t;\gamma,t_o) \geq F(0;\gamma;t_o)\exp\left[\left\{\frac{F'(0;\gamma,t_o) + \frac{\lambda}{|\alpha|}F(0;\gamma,t_o)}{|\alpha|F(0;\gamma,t_o)}\right\}(1-e^{-|\alpha|t}) - \frac{\lambda}{|\alpha|}t\right] \tag{3.40}$$

However, $F(0;\gamma,t_o) = \gamma t_o^2$ and $F'(0;\gamma,t_o) = 2\gamma t_o$. Therefore, if we set $\gamma = 1/t_o^2$ in (3.40) we obtain

$$||u(t)||^2 + [t/t_o + 1]^2 \geq \exp[\chi(t;t_o)], \; 0 \leq t < T \tag{3.41}$$

where

$$\chi(t;t_o) \equiv \frac{1}{|\alpha|}\left[\left(\frac{2}{t_o} + \frac{\lambda(1/t_o^2;t_o)}{|\alpha|}\right)(1-e^{-|\alpha|t}) - \lambda(1/t_o^2;t_o)t\right] \tag{3.42}$$

and

$$\lambda(1/t_o^2;t_o) = 2\left(||\underset{\sim}{u}_1||^2 + \frac{1}{t_o^2} + \omega^2N^2T \sup_{[0,T)} ||\underset{\sim}{M}||_{L(H_+,H_-)}\right) \tag{3.43}$$

We note, in passing, that $\chi(0;t_o) = 0$. For the sake of convenience we now set

$$\epsilon(t_o) = \frac{2}{t_o} + \frac{\lambda(1/t_o^2;t_o)}{|\alpha|}$$

Then

$$\chi'(t;t_o) = \epsilon(t_o)e^{-|\alpha|t} - \lambda(1/t_o^2;t_o) \tag{3.44}$$

From (3.44) it follows immediately that $\chi'(t;t_o) > 0$ for $0 < t < \frac{1}{|\alpha|}\ln\left[\frac{\epsilon(t_o)}{\lambda(1/t_o^2;t_o)}\right]$ provided $\epsilon(t_o) > \lambda(1/t_o^2;t_o)$.

We now take the limit in (3.41) as $t_o \to +\infty$ and obtain

$$||\underset{\sim}{u}(t)||^2 + 1 \geq \exp[\lim_{t_o \to +\infty} \chi(t;t_o)],\ 0 \leq t < T \tag{3.45}$$

But

$$\lim_{t_o \to +\infty} \chi(t;t_o) = \frac{1}{|\alpha|} [\lim_{t_o \to +\infty} \epsilon(t_o)(1-e^{-|\alpha|t}) \tag{3.46}$$

$$- \lim_{t_o \to +\infty} \lambda(1/t_o^2;t_o)]$$

$$= \frac{\bar{\lambda}}{|\alpha|^2}(1-e^{-|\alpha|t}) - \bar{\lambda}t \equiv \bar{\chi}(t)$$

where $\bar{\lambda}$ is given by (3.28c). Also

$$\lim_{t_o \to +\infty} \chi'(t;t_o) = \frac{d}{dt}\bar{\chi}(t) = \bar{\lambda}(\frac{e^{-|\alpha|t}}{|\alpha|} - 1) \tag{3.47}$$

and, therefore,

$$\bar{\chi}'(t) > 0, \quad 0 \leq t < \frac{1}{|\alpha|}\ \ell n(\frac{1}{|\alpha|}) \tag{3.48}$$

if $|\alpha| < 1$. The statement of the theorem now follows with $T = \frac{1}{|\alpha|}\ \ell n(\frac{1}{|\alpha|})$, i.e.,

$$||\underset{\sim}{u}(t)||^2 + 1 \geq \exp(\bar{\chi}(t)),\ 0 \leq t < \frac{1}{|\alpha|}\ \ell n(\frac{1}{|\alpha|}) \tag{3.49}$$

where $\bar{\chi}(t)$, as determined by (3.46_2), is nonnegative and monotonically increasing on $[0, \frac{1}{|\alpha|}\ \ell n(\frac{1}{|\alpha|}))$.

Q.E.D.

Remark In deducing the differential inequality (3.18) we took advantage of the hypothesis that $\alpha < 0$ to drop the expression

$$H(t;\gamma;t_o) \equiv -2\alpha F(\gamma(t+t_o) + 4\int_0^t K(\tau)d\tau) \tag{3.50}$$

from (3.3); if we retain this expression in (3.3) then the upper bound obtained in theorem III.1 may be sharpened somewhat. In order to show this let us note, first of all that, as $\underset{\sim}{u}_o = \underset{\sim}{0}$,

$$\underset{\sim}{u}(t) = \int_o^t \underset{\sim}{u}_\tau(\tau)d\tau,\ 0 \le t < T \tag{3.51}$$

We have, therefore, the estimates

$$\begin{aligned} ||\underset{\sim}{u}(t)||^2 &\le (\int_0^t||\underset{\sim}{u}_\tau(\tau)||d\tau)^2 \\ &\le t\int_0^t||\underset{\sim}{u}_\tau(\tau)||^2 d\tau \\ &\le T\int_0^t||\underset{\sim}{u}_\tau(\tau)||^2 d\tau \end{aligned} \tag{3.52}$$

But

$$\begin{aligned} ||\underset{\sim}{u}(t)||^2 &= F(t;\gamma,t_o) - \gamma(t+t_o)^2 \\ &> F(t;\gamma,t_o) - \gamma(T+t_o)^2, \end{aligned} \tag{3.53}$$

so for any $t \in [0,T)$

$$\begin{aligned} \int_0^t||\underset{\sim}{u}_\tau(\tau)||^2 d\tau &> \frac{1}{T}||\underset{\sim}{u}(t)||^2 \\ &> \frac{1}{T}F(t;\gamma,t_o) - \frac{\gamma}{T}(T+t_o)^2 \end{aligned} \tag{3.54}$$

Therefore, from (3.50) and the assumption that $\alpha < 0$ we obtain the lower bound

$$H(t;\gamma,t_o) = 2|\alpha|F(\gamma(t+t_o) + 2\int_0^t ||\underset{\sim}{u}_\tau(\tau)||^2 d\tau) \tag{3.55}$$

$$> \frac{4|\alpha|}{T} F^2 - 2F(\frac{2|\alpha|\gamma}{T} (T+t_o)^2 - |\alpha|\gamma t_o)$$

From (3.3) (with $\underset{\sim}{u}_o = \underset{\sim}{0}$, $\alpha < 0$), (3.50), and (3.55) we now obtain the differential inequality

$$FF'' - F'^2 \geq - 2F(||\underset{\sim}{u}_1||^2 + \hat{\mu}) + \frac{4|\alpha|F^2}{T} - |\alpha|FF', \tag{3.56}$$

where

$$\hat{\mu} \equiv \gamma[1 - |\alpha|t_o + \frac{2|\alpha|}{T} (T+t_o)^2] \tag{3.57}$$

$$+ \omega N^2 T \sup_{[0,T)} ||\underset{\sim}{M}(t)||_{L(H_+,H_-)} \tag{3.57}$$

By virtue of the same reasoning which led to (3.22) we have the following result: If

$$\tilde{\lambda}(\gamma;t_o) \equiv \frac{2(||\underset{\sim}{u}_1||^2 + \hat{\mu})}{\gamma t_o^2} \tag{3.58}$$

then $F(t;\gamma,t_o)$ satisfies, for all $t \in [0,T)$,

$$FF'' - F'^2 \geq - (\tilde{\lambda}(\gamma;t_o) - \frac{4|\alpha|}{T})F^2 - |\alpha|FF' \tag{3.59}$$

and, as a consequence, we obtain the estimate

$$F(t;\gamma,t_o) \leq e^{\frac{-\tilde{\lambda}}{|\alpha|} t} e^{\frac{4t}{T}} [F(t_1;\gamma,t_o)e^{\frac{\tilde{\lambda}}{|\alpha|} t_1} e^{\frac{-4t_1}{T}}]^{\delta(t)} \tag{3.60}$$

$$\times [F(t_2;\gamma,t_o)e^{\frac{\tilde{\lambda}}{|\alpha|}t_2} e^{\frac{-4t_2}{T}}]^{1-\delta(t)},$$

where $\delta(t)$ is given by (3.25) and $[t_1,t_2] \subseteq [0,T)$ is any closed interval such that $F(t;\gamma;t_o) > 0$, $t_1 \le t \le t_2$. With the assumption that $F(t;\gamma;t_o) > 0$ on $[0,T)$ we may set $t_1 = 0$, $t_2 = T$ in (3.60) and obtain

$$F(t;\gamma,t_o) \le e^{\frac{-\tilde{\lambda}}{|\alpha|} t} e^{\frac{4t}{T}} [\gamma t_o^2]^{\bar{\delta}} [F(T;\gamma,t_o) e^{\frac{\tilde{\lambda}}{|\alpha|} T}]^{1-\bar{\delta}}, \tag{3.61}$$

where $\bar{\delta}(t)$ is given by (3.27). We note, in passing, that we have dropped, from (3.61), the factor $e^{-4(1-\bar{\delta})}$ whose suprėmum on $[0,T)$ is 1. We again set $\gamma = 1/t_o^2$ and take the limit (in (3.61)) as $t_o \to +\infty$. Clearly, (3.28a) and (3.28b) still apply. Also, from (3.58) and (3.57)

$$\lim_{t_o \to +\infty} \tilde{\lambda}(1/t_o^2;t_o) = \bar{\lambda} + \frac{4|\alpha|}{T} \equiv \hat{\lambda}, \tag{3.62}$$

where $\bar{\lambda}$ is defined by (3.28c). Thus, (3.61) yields

$$||\underset{\sim}{u}(t)||^2 \le Ae^{4(1-\bar{\delta})} e^{\frac{\bar{\lambda}}{|\alpha|} g(t)}, \quad 0 \le t < T \tag{3.63}$$

where $g(t)$ is again given by (3.31) while A is defined by (3.30). Our results can be summarized in the following corollary to theorem III.1:

<u>Corollary III.1</u> Let $\underset{\sim}{u} \in N$ be any solution of (2.17), (2.18) with $\underset{\sim}{u}_o = \underset{\sim}{0}$ and $\alpha < 0$. Assume that $\underset{\sim}{M}(0)$ satisfies (3.2a) and (3.2b) and that $T > \frac{1}{|\alpha|}$. Then $||\underset{\sim}{u}(t)||^2$ satisfies the

estimate (3.63) on $[0,T)$ where A is given by (3.30), $\bar{\delta}(t)$ by (3.27), and $g(t) \equiv T(1-\bar{\delta}(t)) - t$ satisfies (i) and (ii) of Theorem II.1.

Remark In contrast to the improved upper bound obtained above, retention of the expression (3.50) in (3.3) does not lead to an improvement over the lower bound obtained in (3.49). This is easily seen as follows: If we retain the expression (3.50) in (3.3) we are led to consider the differential inequality (3.59) in lieu of (3.22). By the "tangent property" for convex functions we again obtain an estimate of the form (3.41) with $\chi(t;t_o)$ replaced by

$$\tilde{\chi}(t;t_o) \equiv \frac{1}{|\alpha|}\Big(\frac{2}{t_o} + \frac{\tilde{\lambda}(1/t_o^2;t_o)}{|\alpha|} - \frac{4}{T}\Big)(1-e^{-|\alpha|t}) \tag{3.64}$$

$$-\Big(\tilde{\lambda}(1/t_o^2;t_o) - \frac{4|\alpha|}{T}\Big)t],$$

where $\tilde{\lambda}$ is given by (3.57) and (3.58). In view of (3.62), however, it is clear that

$$\begin{aligned}\lim_{t_o\to+\infty} \tilde{\chi}(t;t_o) &= \frac{\bar{\lambda}}{|\alpha|^2}(1-e^{-|\alpha|t}) - \bar{\lambda}t \\ &= \bar{\chi}(t) \\ &= \lim_{t_o\to+\infty} \chi(t;t_o)\end{aligned}$$

and, therefore, precisely the same lower bound, i.e. (3.49), is obtained as $t_o \to +\infty$ if we retain the terms in (3.50) in

the differential inequality (3.3).

Case II: $\underset{\sim}{u}_o = 0$ and $\alpha > 0$

In this case the expression (3.50) can not be dropped from the differential inequality (3.3). As $t < T$ and $\alpha > 0$, (3.3) with $\underset{\sim}{u}_o = \underset{\sim}{0}$ implies that

$$FF'' - F'^2 \geq -2F(||\underset{\sim}{u}_1||^2 + \mu) + \alpha FF' - 2\alpha F(\gamma(T+t_o) + 2\int_o^t ||\underset{\sim}{u}_\tau||^2 d\tau) \tag{3.66}$$

In order to proceed further we shall need the following

Lemma Let $\underset{\sim}{u} \in N$ be any solution of (2.17), (2.18) with $\underset{\sim}{u}_o = \underset{\sim}{0}$. Then there exists a real-valued continuous function $h_\alpha(t)$, defined for $0 \leq t < T$, such that

$$\frac{1}{2t} \int_0^t ||\underset{\sim}{u}_\tau||^2 d\tau \leq ||\underset{\sim}{u}_1||^2 + h_\alpha(T), \quad 0 \leq t < T \tag{3.67}$$

Proof: From the identity

$$\underset{\sim}{u}_t = \int_0^t \underset{\sim}{u}_{\tau\tau}\, d\tau + \underset{\sim}{u}_1,$$

and (2.17), we easily obtain (after substituting for $\underset{\sim}{u}_{\tau\tau}$ and using the assumption that $\underset{\sim}{u}_o = \underset{\sim}{0}$)

$$\underset{\sim}{u}_t = \underset{\sim}{u}_1 + \alpha \underset{\sim}{u} + \int_0^t \underset{\sim}{L}\underset{\sim}{u}(\tau)d\tau - \int_0^t\int_0^\tau \underset{\sim}{M}(\tau-\sigma)\underset{\sim}{u}(\sigma)d\sigma d\tau \tag{3.68}$$

Thus,

$$||\underset{\sim}{u}_t|| \le ||\underset{\sim}{u}_1|| + \alpha||\underset{\sim}{u}(t)|| + \int_0^t ||\underset{\sim}{L}||_{L(H_+,H_-)} ||\underset{\sim}{u}(\tau)||_+ d\tau \quad (3.69)$$

$$+ \int_0^t \int_0^\tau ||\underset{\sim}{M}(t-\sigma)||_{L(H_+,H_-)} ||\underset{\sim}{u}(\sigma)||_+ d\sigma d\tau$$

$$\le ||\underset{\sim}{u}_1|| + \alpha\omega||\underset{\sim}{u}(t)||_+ + t||\underset{\sim}{L}||_{L(H_+,H_-)} \sup_{[0,T)} ||\underset{\sim}{u}(\tau)||_+$$

$$+ \frac{t^2}{2} \sup_{[0,t)} ||\underset{\sim}{M}(t)||_{L(H_+,H_-)} \sup_{[0,T)} ||\underset{\sim}{u}(\tau)||_+$$

$$\le ||\underset{\sim}{u}_1|| + p_\alpha(t) \sup_{[0,t)} ||\underset{\sim}{u}(\tau)||_+$$

where

$$p_\alpha(t) \equiv \alpha\omega + t||\underset{\sim}{L}||_{L(H_+,H_-)} + \frac{t^2}{2} \sup_{[0,T)} ||\underset{\sim}{M}(t)||_{L(H_+,H_-)} \quad (3.70)$$

Clearly $p_\alpha(t) < p_\alpha(T)$, for all $t \in [0,T)$ and, as $\underset{\sim}{u} \in N$

$$||\underset{\sim}{u}_t|| \le ||\underset{\sim}{u}_1|| + Np_\alpha(T), \; 0 \le t < T \quad (3.71)$$

Therefore,

$$\int_0^t ||\underset{\sim}{u}_\tau||^2 d\tau \le 2t(||\underset{\sim}{u}_1||^2 + N^2 p_\alpha^2 (T)), \quad 0 \le t < T \quad (3.72)$$

and the lemma follows with

$$h_\alpha(t) = N^2 p_\alpha^2(t) \quad (3.73)$$

If we combine (3.66) with (3.67) we obtain the differential inequality

$$FF'' - F'^2 \geq -2F(||\underset{\sim}{u}_1||^2 + \tilde{\mu}) + \alpha FF' \tag{3.74}$$

where $\tilde{\mu} > 0$ is defined by

$$\tilde{\mu} = \mu + \alpha[\gamma(T+t_o) + 4T(||\underset{\sim}{u}_1||^2 + h_\alpha(T))] \tag{3.75}$$

Choosing

$$\lambda^* = \lambda^*(\gamma;t_o) = \frac{2(||\underset{\sim}{u}_1||^2 + \tilde{\mu})}{\gamma t_o^2} \tag{3.76}$$

we have

$$FF'' - F'^2 \geq -\lambda^*(\gamma;t_o)F^2 + \alpha FF', \quad 0 \leq t < T, \tag{3.77}$$

and upper and lower bounds for $||\underset{\sim}{u}(t)||^2$ may be derived from the differential inequality (3.77) in a manner analogous to that followed in analyzing (3.22); the essential difference between (3.22) and (3.77) is of course, not the difference in complexity between the respective coefficients of the quantity F^2, but rather the simple difference in the signs of the coefficients of the term FF'.

If we apply Jensen's inequality to (3.77), taking $[0,T)$ as the relevant interval, we obtain

$$F(t;\gamma,t_o) \leq e^{\frac{\lambda^* t}{\alpha}}[\gamma t_o^2]^{\delta^*}[F(T;\gamma,t_o)e^{\frac{-\lambda^*}{\alpha}T}]^{1-\delta^*}, \tag{3.78}$$

where

$$\delta^*(t) = (e^{\alpha t} - e^{\alpha T})/(1-e^{\alpha T}), \quad 0 \leq t < T \tag{3.79}$$

As $\alpha > 0$,

$$\delta^{*\prime}(t) = \frac{\alpha e^{\alpha t}}{1-e^{\alpha T}} < 0, \quad 0 \le t < T \tag{3.80a}$$

so

$$|\delta^{*\prime}(t)| = \alpha e^{\alpha t}/(e^{\alpha T} - 1), \quad 0 \le t < T \tag{3.80b}$$

In (3.78) we again use the device of first setting $\gamma = 1/t_o^2$ and then taking the limit, on both sides of the inequality, as $t_o \to +\infty$. Since

$$\begin{aligned}\lim_{t_o \to +\infty} \lambda^*(1/t_o^2;t_o) &= \bar{\lambda} + 8\alpha T(||\underset{\sim}{u}_1||^2 + h_\alpha(T)) \\ &\equiv \bar{\lambda}^*,\end{aligned} \tag{3.81}$$

with $\bar{\lambda}$ defined by (3.28c), we obtain

$$||\underset{\sim}{u}(t)||^2 \le Be^{\frac{\bar{\lambda}^*}{\alpha} f(t)}, \quad 0 \le t < T \tag{3.82}$$

where $f(t) \equiv t - (1-\delta^*)T$ and

$$B \equiv \sup_{[0,T)} (\omega^2 N^2 + 1)^{1-\delta^*(t)} \tag{3.83}$$

<u>Remark</u> If we choose $Q > 0$ so large that

$$\omega^2 N^2 + 1 \le Qe^{\frac{\bar{\lambda}^*}{\alpha} T}, \tag{3.84}$$

then we obtain (from (3.78)) the estimate

$$||\underset{\sim}{u}(t)||^2 \le Q^{1-\delta^*} e^{\frac{\bar{\lambda}^*}{\alpha} t}, \quad 0 \le t < T \tag{3.85}$$

If, of course, there exists $T_o > 0$ such that

$$(\omega^2 N^2 + 1) e^{-\frac{\bar{\lambda}^*}{\alpha} T_o} < 1 \tag{3.86}$$

then (3.85) may be replaced by the much simpler estimate

$$||\underset{\sim}{u}(t)||^2 \le e^{\frac{\bar{\lambda}^*}{\alpha} t}, \quad 0 \le t < T_o \tag{3.87}$$

We now return to (3.82). Directly from the definition of f(t) (and (3.80a))

$$\dot{f}'(t) = 1 + T\dot{\delta}^{*'}(t) = 1 - T|\delta^{*'}(t)|, \; 0 \le t < T \tag{3.88}$$

Therefore,

$$\dot{f}'(t) \begin{Bmatrix} < 0 \\ = 0 \\ > 0 \end{Bmatrix} \text{ if and only if } |\delta^{*'}(t)| \begin{Bmatrix} > \frac{1}{T} \\ = \frac{1}{T} \\ < \frac{1}{T} \end{Bmatrix} \tag{3.89}$$

But by (3.80b)

$$|\delta^{*'}(t)| < 1/T \iff \alpha e^{\alpha t} < \frac{1}{T}(e^{\alpha T} - 1) \tag{3.90}$$

and the latter inequality is satisfied if and only if

$$t < \frac{1}{\alpha} \ln\left[\frac{e^{\alpha T} - 1}{\alpha T}\right] \tag{3.91}$$

(Note that $m(t) \equiv e^t - t - 1 > 0$ for all $t > 0$ so that, in

particular, $e^{\alpha T} - 1 \geq \alpha T$ for $\alpha > 0$). As $f(0) = 0$ and $\lim_{t \to T^-} f(t) = 0$ we can summarize our results in the following theorem:

<u>Theorem III.3</u> Let $\underset{\sim}{u} \in N$ be any solution of (2.17), (2.18) with $\underset{\sim}{u}_o = 0$ and $\alpha > 0$. Assume that $\underset{\sim}{M}(0)$ satisfies (3.2a) and (3.2b). Then for all $T > 0$, $||\underset{\sim}{u}||$ satisfies (3.82) where $\bar{\lambda}^*$ and B are defined, respectively, by (3.81) and (3.83), and $f(t)$ is a nonnegative real-valued function which satisfies

(i') $f(0) = 0, \lim_{t \to T^-} f(t) = 0$

(ii') $f(t)$ is (strictly) monotonically $\left\{\begin{matrix}\text{increasing}\\ \text{decreasing}\end{matrix}\right\}$

for $\left\{\begin{matrix} t < \\ t > \end{matrix}\right\} \frac{1}{\alpha} \ell n\left[\frac{e^{\alpha T} - 1}{\alpha T}\right]$

To close out our study of the case $\underset{\sim}{u}_o = 0$, $\alpha > 0$ we now integrate the differential inequality (3.7) according to the "tangent property" of convex functions and we obtain the estimate

$$F(t;\gamma,t_o) \geq \gamma t_o^2 \exp\left[\left\{\frac{2\gamma t_o - \frac{\lambda^*}{\alpha}\gamma t_o^2}{-\alpha\gamma t_o^2}\right\}(1-e^{\alpha t}) + \frac{\lambda^*}{\alpha} t\right] \tag{3.92}$$

which, with $\gamma = 1/t_o^2$, $\lambda^* = \lambda^*(1/t_o^2;t_o)$, reduces to

$$||\underset{\sim}{u}(t)||^2 + \left(\frac{t}{t_o} + 1\right)^2 \geq \exp\left[\left\{\frac{\lambda^*}{\alpha^2} - \frac{2}{\alpha t_o}\right\}\cdot(1-e^{\alpha t}) + \frac{\lambda^*}{\alpha} t\right] \tag{3.93}$$

Were we to follow the arguments previously employed we would, at this point, take the limit in (3.93) as $t_o \to +\infty$. If we

proceed in this fashion, however, we obtain

$$||\underset{\sim}{u}(t)||^2 + 1 \geq \exp\left[\frac{\bar{\lambda}^*}{\alpha}\left(\frac{1}{\alpha}(1-e^{\alpha t}) + t\right)\right] \tag{3.94}$$

But $\tilde{\chi}(t) \equiv \frac{1}{\alpha}(1-e^{\alpha t}) + t$ satisfies $\tilde{\chi}(0) = 0$ and

$$\tilde{\chi}'(t) = -e^{\alpha t} + 1 < 0, \quad 0 < t < T, \tag{3.95}$$

if $\alpha > 0$. Thus

$$\exp\left[\frac{\bar{\lambda}^*}{\alpha}\left(\frac{1}{\alpha}(1-e^{\alpha t}) + t\right]\right. \leq 1, \ 0 \leq t < T$$

which means, of course, that the estimate (3.94), obtained from (3.93) by letting $t_o \to +\infty$, is without any value as far as obtaining a lower bound on $||\underset{\sim}{u}(t)||^2$ goes. An exponentially increasing lower bound could be obtained for $||\underset{\sim}{u}(t)||^2$ from (3.93) if we could find t_o (real and nonnegative) such that

$$t_o \lambda^*(1/t_o^2; t_o) = 2\alpha, \tag{3.96}$$

however, a little algebra shows that this equation possesses only pure imaginary roots. In fact, if we rewrite the estimate (3.92) in the form

$$||\underset{\sim}{u}(t)||^2 + \gamma(t+t_o)^2 \geq \gamma t_o^2 \exp\left[\left\{\frac{\lambda^*}{\alpha^2} - \frac{2}{\alpha t_o}\right\}\cdot(1-e^{\alpha t}) + \frac{\lambda^*}{\alpha}t\right] \tag{3.97}$$

where $\lambda^* = \lambda^*(\gamma; t_o)$ is defined by (3.75), (3.76), and (3.17), then it is not possible to determine $\gamma = \gamma(t_o)$ such that

$$t_o\lambda^*(\gamma(t_o);\ t_o) = 2\alpha \tag{3.98}$$

with $\gamma(t_o)$ real and nonnegative. It is worthwhile, however, to examine the function

$$J(t;\gamma,t_o) \equiv \left(\frac{\lambda^*(\gamma;t_o)}{\alpha^2} - \frac{2}{\alpha t_o}\right)\cdot(1-e^{\alpha t}) + \frac{\lambda^*(\gamma;t_o)t}{\alpha} \tag{3.99}$$

Clearly, $J(0;\gamma,t_o) = 0$ for arbitrary nonnegative constants γ, t_o. Also

$$J'(t;\gamma,t_o) = \left(\frac{2}{t_o} - \frac{\lambda^*(\gamma;t_o)}{\alpha}\right)e^{\alpha t} + \frac{\lambda^*(\gamma;t_o)}{\alpha} \tag{3.100}$$

from which, by the definition of λ^*, it follows that

$$\left(\frac{\alpha\gamma t_o^2}{2}\right) J'(t;\gamma,t_o) = (k_1 + k_2\gamma)(1-e^{\alpha t}) + \alpha\gamma t_o \tag{3.101}$$

where

$$k_1 = ||\underset{\sim}{u}_1||^2(1 + 4\alpha T) + \bar{\mu} + 4\alpha T h_\alpha(T) \tag{3.102a}$$

$$k_2 = 1 + \alpha T \tag{3.102b}$$

Thus, if we choose

$$t_o = t_{o,\gamma} \equiv \frac{(k_1+k_2\gamma)}{\alpha\gamma}(e^{\alpha T} - 1),\ \gamma > 0 \tag{3.103}$$

then $J'(t;\gamma,t_{o,\gamma}) > 0$ for all t, $0 \le t \le T$, and each real $\gamma > 0$, and we can state the following result:

<u>Theorem III.4</u> Let $\underset{\sim}{u} \in N$ be any solution of (2.17), (2.18)

with $\underset{\sim}{u}_o = 0$ and $\alpha > 0$ and assume that $\underset{\sim}{M}(0)$ satisfies (3.2a), (3.2b). Then for any $T > 0$ and each real $\gamma > 0$

$$||\underset{\sim}{u}(t)||^2 + \gamma(t+t_{o,\gamma})^2 \geq \gamma t_{o,\gamma}^2 \exp[J(t;\gamma,t_{o,\gamma})],\ 0 \leq t < T, \quad (3.104)$$

where $t_{o,\gamma}$ is defined by (3.102a), (3.102b), and (3.103) and $J(t;\gamma,t_{o,\gamma})$, defined by (3.99) with $t_o = t_{o,\gamma}$, is nonnegative and strictly monotonically increasing on $[0,T)$.

The results obtained in cases I and II did not involve any hypotheses concerning the sign of the initial energy $E(0)$; as we assumed $\underset{\sim}{u}_o = \underset{\sim}{0}$ in both cases, $E(0) = \frac{1}{2}||\underset{\sim}{u}_1||^2 > 0$ if $\underset{\sim}{u}_1 \neq \underset{\sim}{0}$. In the cases considered below we remove the restriction that $\underset{\sim}{u}_o = \underset{\sim}{0}$.

<u>Case III</u>: $\underset{\sim}{u}_o \neq \underset{\sim}{0}$, $\alpha > 0$, and $\beta(t) = 0$, $0 \leq t < T$.

In this case (provided we use the fact that $\alpha < 0$ to delete the term $H(t;\gamma,t_o)$ defined by (3.50)) the inequality (3.3) reduces to

$$FF'' - F'^2 \geq -2F(||\underset{\sim}{u}_1||^2 - \langle \underset{\sim}{u}_0, \underset{\sim}{L}\underset{\sim}{u}_0\rangle + \mu) \quad (3.105)$$

$$- |\alpha| FF'$$

with μ given by (3.17). We now assume that the initial data $\underset{\sim}{u}_0$, $\underset{\sim}{u}_1$ satisfies

$$||\underset{\sim}{u}_1||^2 - \langle \underset{\sim}{u}_0, \underset{\sim}{L}\underset{\sim}{u}_0\rangle < -\bar{\mu} \quad (3.106)$$

where $\bar{\mu} = \omega N^2 T \sup_{[0,T)} ||\underset{\sim}{M}(t)||_{\mathcal{L}(H_+,H_-)}$. Taking $\gamma = 0$ in (3.105) we obtain

$$F(t)F''(t) - [F'(t)]^2 \geq -|\alpha|F(t)F'(t), \quad 0 \leq t < T, \tag{3.107}$$

where $F(t) = ||\underset{\sim}{u}(t)||^2$. Jensen's inequality then yields the upper bound

$$||\underset{\sim}{u}(t)||^2 \leq ||\underset{\sim}{u}_o||^{2\bar{\delta}} \, ||\underset{\sim}{u}(T)||^{2(1-\bar{\delta})}, \quad 0 \leq t < T, \tag{3.108a}$$

where $\bar{\delta}(t)$ is given by (3.27). We note that the hypothesis that $\underset{\sim}{u} \in N$, and (3.108), imply that there exists $R > 0$ such that

$$||\underset{\sim}{u}(t)||^2 \leq R^{1-\bar{\delta}} ||\underset{\sim}{u}_o||^{2\bar{\delta}}, \quad 0 \leq t < T \tag{3.108b}$$

However, as (3.106) can not be valid for $||\underset{\sim}{u}_o||$ sufficiently small, (3.108b) represents only an upper bound on $||\underset{\sim}{u}(t)||$ in terms of $||\underset{\sim}{u}_o||$ and not a stability estimate. A better result is found by integrating (3.107) according to the "tangent property" of convex function; in fact, directly from (3.40) with $\lambda = 0$ and $F(t;\gamma,t_o)$ replaced by $F(t) \equiv ||\underset{\sim}{u}(t)||^2$ we obtain

$$||\underset{\sim}{u}(t)||^2 \geq ||\underset{\sim}{u}_o||^2 \exp\left[\frac{2\langle \underset{\sim}{u}_1, \underset{\sim}{u}_o\rangle}{|\alpha| \, ||\underset{\sim}{u}_o||^2} (1-e^{-|\alpha|t})\right], \quad 0 \leq t < T \tag{3.109}$$

Directly from the estimate (3.109) is obvious that if either $\langle \underset{\sim}{u}_o, \underset{\sim}{u}_1\rangle = 0$ or $\underset{\sim}{u}_1 = \underset{\sim}{0}$ (and $\langle \underset{\sim}{u}_o, \underset{\sim}{L}\underset{\sim}{u}_o\rangle > \bar{\mu}$) then

$||\underset{\sim}{u}(t)||^2 \geq ||\underset{\sim}{u}_o||^2$ for all $t \in [0,T)$. On the other hand, if $\langle \underset{\sim}{u}_1, \underset{\sim}{u}_o \rangle > 0$, then on $[0,T)$ $||\underset{\sim}{u}(t)||^2$ is bounded below by a monotonically increasing exponential function of t. Finally if $\langle \underset{\sim}{u}_o, \underset{\sim}{u}_1 \rangle < 0$ then $||\underset{\sim}{u}(t)||^2$ can not decay any faster than a monotonically decreasing exponential function of t. Our results are summarized as

Theorem III.5 Let $\underset{\sim}{u} \in N$ be any solution of (2.17), (2.18) with $\underset{\sim}{u}_o \neq \underset{\sim}{0}$, $\alpha < 0$, and $\beta(t) \equiv 0$ on $[0,T)$. Assume that $\underset{\sim}{M}(0)$ satisfies (3.2a) and (3.2b). Then

(A) If the initial data satisfy (3.106) $||\underset{\sim}{u}(t)||$ is bounded above by $||\underset{\sim}{u}_o||$ according to (3.108b), for all $t \in [0,T)$

(B) If the initial data satisfy (3.106) then there exists $K(\alpha)$ such that for all t, $0 \leq t < T$,

$$||\underset{\sim}{u}(t)||^2 \geq ||\underset{\sim}{u}_o||^2 \exp[K(\alpha)(1-e^{-|\alpha|t})], \tag{3.110}$$

where for each real α, $K(\alpha)$ is real-valued and

(i) $K(\alpha) = 0$ if either $\underset{\sim}{u}_1 = 0$ or $\langle \underset{\sim}{u}_o, \underset{\sim}{u}_1 \rangle = 0$

(ii) $K(\alpha) > 0$ if $\langle \underset{\sim}{u}_o, \underset{\sim}{u}_1 \rangle > 0$

(iii) $K(\alpha) < 0$ if $\langle \underset{\sim}{u}_o, \underset{\sim}{u}_1 \rangle < 0$

and

(iv) $|K(\alpha)| \to 0$ as $|\alpha| \to \infty$.

Remark The case $\underset{\sim}{\mu}_o \neq \underset{\sim}{0}$, $\alpha > 0$, and $\beta(t) \equiv 0$ can be treated in the same manner as Case III; in fact, from (3.74) (which was derived under the assumption that $\underset{\sim}{u}_o = \underset{\sim}{0}$ with $\alpha > 0$) we can write down immediately the differential inequality

$$FF'' - F'^2 \geq -2F(||\underset{\sim}{u}_1||^2 - \langle \underset{\sim}{u}_o, L\underset{\sim}{u}_o \rangle + \tilde{\mu}) + \alpha FF' \tag{3.111}$$

for the case where $\underset{\sim}{\mu}_o \neq 0$, $\alpha > 0$, but $\beta(t) \equiv 0$; in (3.111) $\tilde{\mu}$ is defined by (3.75) and (3.17). Suppose we set $\gamma = 0$; then if the initial data satisfy

$$(1 + 4\alpha T)||\underset{\sim}{u}_1||^2 - \langle \underset{\sim}{u}_o, \underset{\sim}{L}\underset{\sim}{u}_o \rangle \leq -(\bar{\mu} + 4\alpha Th_\alpha(T))$$

the above differential inequality reduces to

$$F(t)F''(t) - [F'(t)]^2 \geq \alpha F(t)F'(t), \quad 0 \leq t < T, \tag{3.112}$$

where $F(t) = ||\underset{\sim}{u}(t)||^2$. We leave the integration of (3.112) and the analysis of the resulting estimates on $||u(t)||^2$ to the reader and turn, instead, to consider a case where both $\underset{\sim}{u}_o \neq 0$ and $\beta(t) \not\equiv 0$.

<u>Case IV</u> $\underset{\sim}{\mu}_o \neq \underset{\sim}{0}$, $\beta(t) \not\equiv 0$, $\alpha < 0$ and $\beta(0) > 0$

In this case (3.3) is easily seen to imply that

$$\begin{aligned} FF'' - F'^2 &\geq -2F(2E(0) + \mu) - |\alpha|FF' \\ &+ 2F(2\int_0^t \dot{\beta}(\tau)\langle \underset{\sim}{u}, \underset{\sim}{u}_o \rangle d\tau - \beta(t)\langle \underset{\sim}{u}, \underset{\sim}{u}_o \rangle) \\ &+ 4F\beta(0)||\underset{\sim}{u}_o||^2 \\ &= -2F(2E(0) - 2\beta(0)||\underset{\sim}{u}_o||^2 + \mu) - |\alpha|FF' \\ &+ 2F(2\int_0^t \dot{\beta}(\tau)\langle \underset{\sim}{u}, \underset{\sim}{u}_o \rangle d\tau - \beta(t)\langle \underset{\sim}{u}, \underset{\sim}{u}_o \rangle) \end{aligned} \tag{3.113}$$

In order to proceed further we must bound from below the third expression on the right-hand side of the differential inequality (3.113_2); this is accomplished by the following lemma:

<u>Lemma</u> Suppose that $\dot{\beta}(t)$ is bounded on $[0,T)$ for each fixed T, $0 < T < \infty$. Then there exists a constant $C > 0$ such that for all $t \in [0,T)$

$$2\int_0^t \dot{\beta}(\tau)<\underset{\sim}{u},\underset{\sim}{u}_o>d\tau - \beta(t)<\underset{\sim}{u},\underset{\sim}{u}_o> \ \geq - c||\underset{\sim}{u}_o|| \tag{3.114}$$

<u>Proof</u> We set $\rho = \sup\limits_{[0,T)} |\dot{\beta}(t)| < \infty$. Then

$$\left|\int_0^t \dot{\beta}(\tau)<\underset{\sim}{u}(\tau), \underset{\sim}{u}_o>d\tau\right| = \left|<\int_0^t \dot{\beta}(\tau)\underset{\sim}{u}(\tau)d\tau, \underset{\sim}{u}_o>\right| \tag{3.15}$$

$$\leq (\int_0^t \left|\dot{\beta}(\tau)\right| ||\underset{\sim}{u}(\tau)||d\tau)||\underset{\sim}{u}_o||$$

$$\leq \rho(\int_0^T||\underset{\sim}{u}(\tau)||d\tau)||\underset{\sim}{u}_o||$$

$$\leq \rho\omega NT||\underset{\sim}{u}_o||$$

so

$$\int_0^t \dot{\beta}(\tau)<\underset{\sim}{u},\underset{\sim}{u}_o>d\tau \geq - \rho\omega NT||\underset{\sim}{u}_o||, \quad 0 \leq t < T \tag{3.116}$$

Also

$$|\beta(t)<\underset{\sim}{u},\underset{\sim}{u}_o>| \leq |\beta(t)|\cdot|<\underset{\sim}{u},\underset{\sim}{u}_o>| \tag{3.117}$$

$$\leq \omega N|\beta(t)|\cdot||\underset{\sim}{u}_o||$$

$$\leq \omega N \left| \int_0^t \dot{\beta}(\tau)d\tau + \beta(0)\right| ||\underset{\sim}{u}_o||$$

$$\leq \omega N(\rho T + \beta(0))||\underset{\sim}{u}_o||$$

so

$$- \beta(t)<\underset{\sim}{u},\underset{\sim}{u}_o> \geq - \omega N(\rho T + \beta(0))||\underset{\sim}{u}_o||, \quad 0 \leq t < T \tag{3.118}$$

Combining (3.11_2), (3.116), and (3.118) we obtain (3.114) with

$$C = \omega N(3\rho T + \beta(0)) > 0 \tag{3.119}$$

We now return to (3.113_2); in view of the last lemma this latter inequality implies that

$$FF'' - F'^2 \geq - 2F(||\underset{\sim}{u}_1||^2 + \textstyle\sum(\underset{\sim}{u}_o) + \mu) - |\alpha|FF' \tag{3.120}$$

where $\sum: H_+ \to R^+$ is defined by

$$\textstyle\sum(\underset{\sim}{w}) = 2\beta(0)||\underset{\sim}{w}||(\frac{C}{2\beta(0)} - ||\underset{\sim}{w}||) - <\underset{\sim}{w},\underset{\sim\sim}{Lw}> \tag{3.121}$$

for any $\underset{\sim}{w} \in H_+$. If we set $\gamma = 0$ then (3.120) easily reduces to

$$F(t)F''(t) - [F'(t)]^2 \geq - 2F(t)(||\underset{\sim}{u}_1||^2 + \textstyle\sum(\underset{\sim}{u}_o) + \bar{\mu}) \tag{3.122}$$

$$- |\alpha|F(t)$$

with $F(t) = ||\underset{\sim}{u}(t)||^2$ and $\bar{\mu} = \omega N^2 \sup_{[0,T)} ||\underset{\sim}{M}(t)||_{L(H_+,H_-)}$

and we have the following simple result:

<u>Theorem III.6</u> Let $\underset{\sim}{u} \in N$ be any solution of (2.17), (2.18) where $\underset{\sim}{\mu}_o \neq 0$, $\beta(t) \not\equiv 0$, $\alpha < 0$, and $\beta(0) > 0$. Assume that $\underset{\sim}{M}(0)$ satisfies (3.2a), (3.2b) and that $\dot{\beta}(t)$ is bounded for

$0 \le t < T$. Then if the initial data satisfy

$$||\underset{\sim}{u}_1||^2 + \textstyle\sum(\underset{\sim}{u}_o) \le -\bar{\mu}, \tag{3.123}$$

where $\sum$ is defined by (3.121), $||\underset{\sim}{u}(t)||$ satisfies the estimates (3.108) and (3.109). In particular, if $\underset{\sim}{u}_1 = 0$ and $\sum(\underset{\sim}{u}_o) \le -\bar{\mu}$ then $||\underset{\sim}{u}(t)||^2 \ge ||\underset{\sim}{u}_o||^2$ for all t, $0 \le t < T$.

<u>Remark</u> We leave to the reader the consideration of the other cases possible when $\underset{\sim}{u}_o \ne \underset{\sim}{0}$ and $\beta(t) \not\equiv 0$, e.g., $\alpha < 0$ and $\beta(0) \le 0$; the stability and growth estimates which apply in these situations may easily be derived by suitably modifying the last lemma and making use of the basic differential inequalities derived for the previous cases.

4. <u>Examples of Growth Estimates for Electric Displacement Fields in Holohedral Isotropic Dielectrics</u>.

In order to apply the results of the previous section to solutions of the initial-boundary value problem (2.1), (2.9), (2.10)(associated with the constitutive relations (1.22a), (1.22b)) we must first delineate the forms assumed by the basic hypotheses (3.2a), (3.2b). In other words, for the operator $\underset{\sim}{M}(t)$, which is defined by (2.14b), we wish to examine the implications of the condition that

$$-\langle \underset{\sim}{v}, \hat{\underset{\sim}{M}}(0)\underset{\sim}{v}\rangle_H \ge \kappa||\underset{\sim}{v}||^2_{H_+} \tag{4.1a}$$

with

$$\kappa \geq \omega T \sup_{[0,T)} ||\hat{\underset{\sim}{M}}_t||_{L(H_+,H_-)}, \tag{4.1b}$$

where the Hilbert spaces H, H_+ are defined to be the completions of $C_o^\infty(\Omega)$ with respect to the norms induced by the inner products (2.11) and (2.12), respectively, while H_- is the completion of $C_o^\infty(\Omega)$ with respect to the norm (2.13); Ω is, of course, a bounded region in R^3 with smooth boundary $\partial\Omega$ and ω is the embedding constant associated with the inclusion map $i: H_+ \to H$. From (2.14b) and (2.11) we easily compute

$$\begin{aligned} \langle \underset{\sim}{v}, \hat{\underset{\sim}{M}}(0)\underset{\sim}{v}\rangle_H &= -\int_\Omega (\hat{\underset{\sim}{M}}(0)\underset{\sim}{v})_i v_i d\underset{\sim}{x} \\ &= -b_o\ddot{\Psi}(0)\int_\Omega \delta_{ij}v_iv_j d\underset{\sim}{x} \\ &+ \frac{b_o}{a_o}\Phi(0)\int_\Omega \delta_{ik}\delta_{j\ell}\frac{\partial^2 v_k}{\partial x_j \partial x_\ell} v_i d\underset{\sim}{x} \\ &= -b_o\ddot{\Psi}(0)||\underset{\sim}{v}||_H^2 + \frac{b_o}{a_o}\Phi(0)\int_\Omega \delta_{ik}\delta_{j\ell}\frac{\partial^2 v_k}{\partial x_j\partial x_\ell} v_i d\underset{\sim}{x} \end{aligned} \tag{4.2}$$

for any $\underset{\sim}{v} \in H_+$. But if $\underset{\sim}{v} \in H_+$ then

$$\begin{aligned} \int_\Omega \delta_{ik}\delta_{j\ell}\frac{\partial^2 v_k}{\partial x_j\partial x_\ell} v_i d\underset{\sim}{x} &= \int_\Omega \delta_{j\ell} v_k \frac{\partial^2 v_k}{\partial x_j \partial x_\ell} d\underset{\sim}{x} \\ &= -\int_\Omega \delta_{j\ell}\frac{\partial v_k}{\partial x_j}\frac{\partial v_k}{\partial x_\ell} d\underset{\sim}{x} \\ &\equiv -||\underset{\sim}{v}||_{H_+}^2, \end{aligned} \tag{4.3}$$

where we have used integration by parts together with the fact that $\underset{\sim}{v}$ vanishes on $\partial\Omega$[2]. Thus

(2) This follows from the definition of H_+ and a standard trace theorem.

$$- \langle \underset{\sim}{v}, \underset{\sim}{\hat{M}}(0)\underset{\sim}{v}\rangle = - b_o \ddot{\Psi}(0) ||\underset{\sim}{v}||_H^2 - \frac{b_o}{a_o} \Phi(0) ||\underset{\sim}{v}||_{H_+}^2 \tag{4.4}$$

$$\geq - b_o(\omega^2 |\ddot{\Psi}(0)| + \frac{1}{a_o} \Phi(0)) ||\underset{\sim}{v}||_{H_+}^2$$

Therefore, (4.1a), (4.1b) will be satisfied if

$$- b_o(\omega^2 |\ddot{\Psi}(0)| + \frac{1}{a_o} \Phi(0)) \geq \kappa \tag{4.5}$$

with $\kappa \geq \omega T \sup_{[0,T)} ||\underset{\sim}{\hat{M}}_t||_{L(H_+,H_-)}$. For the sake of convenience we now set $T(t) = \ddot{\Psi}(t)$. From (2.14b) again we have, for any $\underset{\sim}{v} \in H_+^{(3)}$,

$$(\underset{\sim}{M}_t \underset{\sim}{v})_i = b_o[\dot{T}(t)\delta_{ij}v_j - \frac{\dot{\Phi}(t)}{a_o} \delta_{ik}\delta_{j\ell} \frac{\partial^2 v_k}{\partial x_j \partial x_\ell}] \tag{4.6}$$

so

$$|\langle \underset{\sim}{v}, \underset{\sim}{\hat{M}}_t \underset{\sim}{v}\rangle_H| = |\int_\Omega [\underset{\sim}{\tilde{M}}_t v]_i v_i dx| \tag{4.7}$$

$$= |b_o \dot{T}(t) ||\underset{\sim}{v}||_H^2 - \frac{b_o}{a_o} \dot{\Phi}(t) \int_\Omega \delta_{j\ell} v_k \frac{\partial^2 v_k}{\partial x_j \partial x_\ell} d\underset{\sim}{x}|$$

$$= b_o|\dot{T}(t) ||\underset{\sim}{v}||_H^2 + \frac{1}{a_o} \dot{\Phi}(t) ||\underset{\sim}{v}||_{H_+}^2|$$

$$\leq b_o(\omega^2 |\dot{T}(t)| + \frac{1}{a_o} |\dot{\Phi}(t)|) ||\underset{\sim}{v}||_{H_+}^2$$

It now follows that

$$||\underset{\sim}{\hat{M}}_t||_{L(H_+,H_-)} = \sup_{\underset{\sim}{v} \in H_+} \frac{|\langle \underset{\sim}{v}, \underset{\sim}{\hat{M}}_t \underset{\sim}{v}\rangle|}{||\underset{\sim}{v}||_{H_+}^2} \leq b_o(\omega^2 |\dot{T}(t)| + \frac{1}{a_o} |\dot{\Phi}(t)|) \tag{4.8}$$

(3) We assume that $\dot{T}$ exists and is bounded on [0,T).

for each t, $0 \le t < T$. Thus, (4.1b) will be satisfied if

$$\kappa \ge \omega T b_o(\omega^2 \sup_{[0,T)} |\dot{T}(t)| + \frac{1}{a_o} \sup_{[0,T)} |\dot{\Phi}(t)|) \qquad (4.9)$$

Combining (4.5) and (4.9) we find that a condition which suffices for the simultaneous satisfaction of (4.1a) and (4.1b) is

$$-(\omega^2 |T(0)| + \frac{1}{a_o} \Phi(0)) \ge \omega T(\omega^2 \sup_{[0,T)} |\dot{T}(t)| + \frac{1}{a_o} \sup_{[0,T)} |\dot{\Phi}(t)|) \quad (4.10)$$

Remark It is clear, from (4.10), that the inequality can be satisfied only if $\Phi(0) < 0$ with $|\Phi(0)| > a_o \omega^2 |T(0)|$.

Recall now that $\Phi(t)$ is defined in terms of $\phi(t)$ by (1.14a), (1.14b) while $\Psi(t)$ is defined in terms of $\psi(t)$ by

$$\Psi(t) = \sum_{n=1}^{\infty} (-1)^n \psi^n(t) \qquad (4.11a)$$

$$\psi^1(t) = \frac{1}{b_o} \psi(t)$$

$$\psi^n(t) = \int_0^t \psi^1(t-\tau)\psi^{n-1}(\tau)d\tau, \ n \ge 2 \qquad (4.11b)$$

It is worthwhile, at this point, to recall the following result which has been proven in [6]:

Lemma Let $\phi(t) \in C^1[0,T)$ and assume that the series (1.14a), as well as the derived series, which is obtained by term by

term differentiation, are uniformly convergent on every interval $[0,T-\varepsilon]$, $0 < \varepsilon < T$. If

$$\sup_{[0,T)} |\phi(t)| < a_o/T \tag{4.12}$$

then

$$\text{(i)} \quad \sup_{[0,T)} |\phi(t)| \le F(T) \tag{4.13a}$$

$$\text{(ii)} \quad \sup_{[0,T)} |\dot{\phi}(t)| \le \frac{F(T)}{T}\left[1 + T\,\frac{\sup_{[0,T)} |\dot{\phi}(t)|}{\sup_{[0,T)} |\phi(t)|}\right] \tag{4.13b}$$

where

$$F(T) = \sup_{[0,T)} |\phi(t)|/(a_o - T\sup_{[0,T)} |\phi(t)|) \tag{4.14}$$

<u>Remark</u> Similar results hold for $\sup_{[0,T)} |\psi(t)|$ and $\sup_{[0,T)} |\dot{\psi}(t)|$, of course, under analogous assumptions on $\psi(t)$ and the series (4.11a), e.g., we require that $\sup_{[0,T)} |\psi(t)| < b_o/T$; the constant $F(T)$ appearing in (4.13a), (4.13b) would, in this case, be replaced by

$$G(T) = \sup_{[0,T)} |\psi(t)|/(b_o - T\sup_{[0,T)} |\psi(t)|) \tag{4.15}$$

In recalling the above lemma we have been motivated by a desire to replace the sufficient condition represented by (4.10) by a condition which involves only the basic memory functions $\phi(t)$, $\psi(t)$ specified in the constitutive relations (1.22a), (1.22b). To this end we note that (1.14a), (1.14b)

and (4.11a), (4.11b) imply, respectively, that

$$\Phi(t) + \frac{1}{a_o}\,\phi(t) = -\frac{1}{a_o}\int_0^t \phi(t-\tau)\Phi(\tau)d\tau \tag{4.16a}$$

$$\Psi(t) + \frac{1}{b_o}\,\psi(t) = -\frac{1}{b_o}\int_0^t \psi(t-\tau)\Psi(\tau)d\tau \tag{4.16b}$$

The derivations of (4.16a), (4.16b) depend only on the assumed uniform convergence of the series defining $\Phi(t)$ and $\Psi(t)$. From (4.16a) and (4.16b) we immediately obtain

$$\Phi(0) = -\frac{1}{a_o}\,\phi(0),\ \Psi(0) = -\frac{1}{b_o}\,\psi(0) \tag{4.17}$$

and thus (4.10) can only be satisfied if $\phi(0) > 0$. Directly from (4.16b) we now compute that

$$\dot{\Psi}(t) + \frac{1}{b_o}\,\dot{\psi}(t) = -\frac{1}{b_o}\,\psi(0)\Psi(t) - \frac{1}{b_o}\int_0^t \psi_t(t-\tau)\Psi(\tau)d\tau \tag{4.18a}$$

$$\ddot{\Psi}(t) + \frac{1}{b_o}\,\ddot{\psi}(t) = -\frac{1}{b_o}\,\psi(0)\dot{\Psi}(t) - \frac{1}{b_o}\,\dot{\psi}(0)\Psi(t) - \frac{1}{b_o}\int_0^t \psi_{tt}(t-\tau)\Psi(\tau)d\tau \tag{4.18b}$$

Therefore,

$$\ddot{\Psi}(0) \equiv T(0) = -\frac{1}{b_o}\left(\ddot{\psi}(0) + \psi(0)\dot{\Psi}(0) + \dot{\psi}(0)\Psi(0)\right) \tag{4.19}$$

However, from (4.17) and (4.18a),

$$\dot{\Psi}(0) = -\frac{1}{b_o}\,\dot{\psi}(0) - \frac{1}{b_o}\,\psi(0)\Psi(0) = -\frac{1}{b_o}\,\dot{\psi}(0) + \frac{1}{b_o^2}\,\psi^2(0) \tag{4.20}$$

Combining (4.17_2) and (4.20_2) with (4.19) we have, finally,

$$T(0) = -\frac{1}{b_o}\left(\frac{1}{b_o^2}\psi^3(0) - \frac{2}{b_o}\psi(0)\dot{\psi}(0) + \ddot{\psi}(0)\right) \tag{4.21}$$

The left-hand side of (4.10) now assumes the form

$$\frac{1}{a_o^2}\phi(0) - \frac{\omega^2}{b_o}\left|\frac{1}{b_o^2}\psi^3(0) - \frac{2}{b_o}\psi(0)\dot{\psi}(0) + \ddot{\psi}(0)\right| \tag{4.22}$$

We now turn our attention to the right-hand side of (4.10). Directly from (4.18b) we obtain

$$\dot{T}(t) = -\frac{1}{b_o}\Big(\psi^{(3)}(t) + \psi(0)T(t) + \dot{\psi}(0)\dot{\Psi}(t) + \ddot{\psi}(0)\Psi(t) + \int_0^t \psi_{ttt}(t-\tau)\Psi(\tau)d\tau\Big) \tag{4.23}$$

Also, in view of (4.18b),

$$\sup_{[0,T)}|T(t)| \le \frac{1}{b_o}\Big[\sup_{[0,T)}|\ddot{\psi}(t)| + |\psi(0)|\sup_{[0,T)}|\dot{\psi}(t)| + \big(|\dot{\psi}(0)| + T\sup_{[0,T)}|\ddot{\psi}(t)|\big)\sup_{[0,T)}|\Psi(t)|\Big] \tag{4.24}$$

while, by (4.23),

$$\sup_{[0,T)}|\dot{T}(t)| \le \frac{1}{b_o}\Big[\sup_{[0,T)}|\psi^{(3)}(t)| + \psi(0)\sup_{[0,T)}|T(t)| + \dot{\psi}(0)\sup_{[0,T)}|\dot{\Psi}(t)| + \big(|\ddot{\psi}(0)| + T\sup_{[0,T)}|\psi^{(3)}(t)|\big)\sup_{[0,T)}|\Psi(t)|\Big] \tag{4.25}$$

If we substitute for $\sup\limits_{[0,T)} |\ \ (t)|$ in (4.25) from (4.24) then it is easily seen that we obtain an estimate of the form

$$\sup_{[0,T)} |\dot{T}(t)| \leq A \sup_{[0,T)} |\Psi(t)| + B \sup_{[0,T)} |\dot{\Psi}(t)| + C \tag{4.26}$$

where, in fact, the constants A, B, C are given by

$$A = \frac{1}{b_o}[T \sup_{[0,T)} |\psi^{(3)}(t)| + |\ddot{\psi}(0)| + \frac{|\psi(0)|}{b_o}(|\dot{\psi}(0)| + T \sup_{[0,T)} |\psi^{(2)}(t)|)]$$

$$B = \frac{1}{b_o}[\ |\dot{\psi}(0)| + \frac{\psi^2(0)}{b_o}]$$

$$C = \frac{1}{b_o}[\ \sup_{[0,T)} |\psi^{(3)}(t)| + \frac{1}{b_o}|\psi(0)| \sup_{[0,T)} |\psi^{(2)}(t)|]$$

As a result of the estimate (4.26), the right-hand side of the inequality (4.10) is bounded above by the expression

$$\omega^3 T(A \sup_{[0,T)} |\Psi(t)| + B \sup_{[0,T)} |\dot{\Psi}(t)| + C) + \frac{\omega T}{a_o} \sup_{[0,T)} |\dot{\Phi}(t)|, \tag{4.27}$$

which, in view of the lemma preceding (4.16a), (4.16b), and the subsequent remark, is itself bounded above by

$$\omega^3 T[AG(T) + \frac{BG(T)}{T}(1 + T \frac{\sup\limits_{[0,T)} |\dot{\psi}(t)|}{\sup\limits_{[0,T)} |\psi(t)|}) + C] \tag{4.28}$$

$$+ \frac{\omega F(T)}{a_o}(1 + T \frac{\sup\limits_{[0,T)} |\dot{\phi}(t)|}{\sup\limits_{[0,T)} |\phi(t)|}) \equiv D,$$

provided $\sup_{[0,T)} |\phi(t)| < \frac{a_o}{T}$ and $\sup_{[0,T)} |\psi(t)| < \frac{b_o}{T}$

From (4.28), the definitions of the constants A, B, C, (4.14), and (4.15), it is clear that

$$\mathcal{D} = \mathcal{D}(\omega, T, a_o, b_o, |\psi^{(i)}(0)|, \sup_{[0,T)} |\phi^{(j)}(t)|, \sup_{[0,T)} |\psi^{(k)}(t)|) \qquad (4.29)$$

with $i = 0,1,2$, $j = 0,1$, and $k = 0,1,2,3$. Thus, $\mathcal{D}$ is computable once Ω, $T > 0$, and the constitutive relations (1.22a), (1.22b) are specified. Furthermore (4.10), and hence (4.1a), (4.1b), will be satisfied if

$$\frac{1}{a_o^2}\,\phi(0) - \frac{\omega^2}{b_o}\left|\,\frac{1}{b_o^2}\,\psi^3(0) - \frac{2}{b_o}\,\psi(0)\dot{\psi}(0) + \ddot{\psi}(0)\right| \geq \mathcal{D} \qquad (4.30)$$

We offer below an example of the kind of considerations which are involved in verifying that (4.30) - and hence the condition represented by (4.1a) and (4.1b) - is satisfied. It must be noted that (4.1a), (4.1b) are implied by (4.30) but that, conversely, (4.30) does not represent a necessary condition which must be satisfied if (4.1a), (4.1b) are to be valid; in particular, we have used some very rough estimates in passing from (4.10) to (4.30) and even the former inequality stands as a sufficient (but not necessary) condition as regards the satisfaction of (4.1a) and (4.1b).

Example In the constitutive equations (1.22a), (1.22b) we take

$$\phi(t) = e^{-\kappa t}, \quad \psi(t) = e^{-t} \tag{4.31}$$

where $K > 0$ is arbitrary; for the sake of convenience we set $T = 1$. The region $\Omega \subseteq R^3$ (and hence the embedding constant ω) are left arbitrary at this point as are the constants a_o, b_o. From (4.31) we have

$$\phi(0) = \sup_{[0,1)} |\phi(t)| = 1, \quad \sup_{[0,1)} |\dot{\phi}(t)| = K \tag{4.32a}$$

and

$$\sup_{[0,1)} |\psi^{(k)}(t)| = 1, \; k = 0,1,2,3 \tag{4.32b}$$

$$\psi(0) = \ddot{\psi}(0) = 1, \quad \dot{\psi}(0) = -1 \tag{4.32c}$$

Therefore, the constants A, B, C in (4.26) are given by

$$A = \frac{2}{b_o}(1 + \frac{1}{b_o}), \quad B = C = \frac{1}{b_o}(1 + \frac{1}{b_o}) \tag{4.33}$$

Also, if $a_o > 1$, $b_o > 1$, then from (4.14) and (4.15)

$$F(1) = \frac{1}{a_o - 1}, \; G(1) = \frac{1}{b_o - 1} \tag{4.34}$$

Combining (4.28) and (4.30) with (4.32a) - (4.32c), (4.33), and (4.34) it follows that the operator $\hat{\underset{\sim}{M}}(t)$, which is defined

by (2.14b), (1.14a), (1.14b), (4.11a), (4.11b), and (4.31), will satisfy the fundamental hypotheses (4.1a) and (4.1b) if a_o, b_o, and ω are chosen so as to satisfy

$$\frac{1}{a_o^2} - \frac{\omega(1+K)}{a_o(a_o-1)} > \omega^3 \frac{(b_o+1)}{b_o^2} \cdot \left(\frac{b_o+3}{b_o-1}\right) + \frac{\omega^2}{b_o}\left(\frac{1}{b_o^2} + \frac{2}{b_o} + 1\right) \tag{4.35}$$

As b_o must be restricted to satisfy $b_o > 1$, the right-hand side of (4.35), which we denote as $\sigma(b_o,\omega)$, is clearly positive. Thus, in order for (4.35) to be satisfied for an arbitrary $a_o > 1$, ω must satisfy

$$\omega = \omega_K < \frac{1}{1+K}\left(1 - \frac{1}{a_o}\right) < \frac{1}{1+K} \tag{4.36}$$

If we now choose Ω so that (4.36) is satisfied and define

$$\tilde{\sigma}(a_o, \omega_K) = \frac{1}{a_o^2} - \frac{\omega_K(1+K)}{a_o(a_o-1)}$$

then (4.35) becomes

$$\tilde{\sigma}(a_o, \omega_K) > \sigma(b_o, \omega_K) \tag{4.37}$$

But

$$\lim_{b_o \to +\infty} \sigma(b_o, \omega) = 0 \text{ (for any } \omega > 0) \tag{4.38}$$

and thus it is clear that for an arbitrary $a_0 > 1$ and $\omega = \omega_K$ defined by (4.36), the inequality (4.35) will be satisfied if b_0 is chosen sufficiently large. We summarize our results in the following lemma:

Lemma Consider the holohedral isotropic dielectric material which is defined by the constitutive relations

$$\underset{\sim}{D}(\underset{\sim}{x},t) = a_0\underset{\sim}{E}(\underset{\sim}{x},t) + \int_0^t e^{-K(t-\tau)}\underset{\sim}{E}(\underset{\sim}{x},\tau)d\tau \tag{4.39a}$$

$$\underset{\sim}{H}(\underset{\sim}{x},t) = b_0\underset{\sim}{B}(\underset{\sim}{x},t) + \int_0^t e^{-(t-\tau)}\underset{\sim}{B}(\underset{\sim}{x},\tau)d\tau \tag{4.39b}$$

where $K > 0$ and $a_0 > 1$ are arbitrary and $(\underset{\sim}{x},t) \in \Omega \times [0,1)$ with $\Omega \subseteq R^3$ chosen so that the bedding constant ω, defined by the inclusion map of H_+(the completion of $C_0^\infty(\Omega)$ w.r.t. (2.11) into H, satisfies (4.36). If $\underset{\sim}{D}(\underset{\sim}{x},t) = 0$ $(\underset{\sim}{x},t) \in \partial\Omega \times [0,1)$, then there exists a constant $\Gamma > 1$ such that the operator $\hat{\underset{\sim}{M}}(t)$, defined by (2.14b), satisfies the basic hypotheses (4.1a), (4.1b) whenever $b_0 \geq \Gamma$.

We now close this discussion of holohedral isotropic material dielectrics by offering, below, an example of how some of the theoremes of the last section (which were derived for the abstract integrodifferential system (2.17), (2.18)) may be used to obtain information about the growth behavior of the electric displacement field in the material which is defined by the constitutive relations (4.39a), (4.39b); in these equations $a_0 > 1$ and $K > 0$ are taken to be arbitrary

and $\Omega \subseteq R^3$ and $b_o > 1$ are assumed chosen so as to satisfy the conditions of the Lemma above, i.e., we take

$$b_o = \Gamma \equiv \inf\{b > 1 \mid \sigma(b, \omega_K) < \tilde{\sigma}(a_o, \omega_K)\} \tag{4.40}$$

where ω_K, σ, $\tilde{\sigma}$ are defined by the discussion preceding the lemma. Comparing (2.17) with (2.15), and using (4.17), (4.31), and (4.40) we easily find that

$$\alpha = -\Psi(0) = \frac{1}{b_o}\psi(0) = \frac{1}{\Gamma} \tag{4.41}$$

As $\Gamma > 1$, $0 < \alpha < 1$. The simplest case we can consider in this situation would seem to be Case III of the last section; thus, we shall assume that, in addition to the constitutive relations (4.39a), (4.39b) we have initial conditions specified of the form

$$\underset{\sim}{D}(\underset{\sim}{x},0) = \underset{\sim}{0},\ \underset{\sim}{D}_t(\underset{\sim}{x},0) = \underset{\sim}{D}_1(\underset{\sim}{x}),\quad \underset{\sim}{x} \in \Omega \tag{4.42}$$

as well as the boundary condition

$$\underset{\sim}{D}(\underset{\sim}{x},t) = \underset{\sim}{0},\ (\underset{\sim}{x},t) \in \partial\Omega \times [0,1) \tag{4.43}$$

The displacements our theorems apply to must lie in the class N, i.e., they should satisfy

$$\sup_{0 \le t < 1} \left(\int_\Omega \frac{\partial D_i(\underset{\sim}{x},t)}{\partial x_j} \frac{\partial D_i(\underset{\sim}{x},t)}{\partial x_j}\, d\underset{\sim}{x}\right)^{\frac{1}{2}} < N \tag{4.44}$$

for some $N > 0$. Directly from theorem III.3 we then have the following result

Theorem IV.1 Let $\underset{\sim}{D}(\underset{\sim}{x},t)$, $(\underset{\sim}{x},t) \in \Omega \times [0,1)$ be any solution of (2.1) (subject to (4.42) and (4.43)) which satisfies (4.44) for some $N > 0$, where we assume that $\Phi(t)$, $\Psi(t)$ are determined, respectively, by (4.31_1), (1.14a), (1.14b) and (4.31_2), (4.11a), (4.11b); we also assume that $a_o > 1$, $K > 0$ are arbitrary while Ω is such that the corresponding embedding constant ω satisfies (4.36) and $b_o = \Gamma$ is defined by (4.40). Then there exists $\Lambda = \Lambda(\Gamma)$ with $\Lambda(\Gamma) > 0$ and $\Lambda(\Gamma) \to +\infty$ as $\Gamma \to +\infty$ such that

$$\int_\Omega \underset{\sim}{D}_i(\underset{\sim}{x},t)\underset{\sim}{D}_i(\underset{\sim}{x},t)d\underset{\sim}{x} \le Be^{\Lambda(\Gamma)f(t)}, \quad 0 \le t < 1 \tag{4.45}$$

where

$$B = \sup_{0 \le t < 1} (\omega_K^2 N^2 + 1)^{1-\delta^*(t)} \tag{4.46a}$$

$$\delta^*(t) = (e^{t/\Gamma} - e^{1/\Gamma})/(1-e^{1/\Gamma}), \quad 0 \le t < 1 \tag{4.46b}$$

$$f(t) = t - (1-\delta^*), \quad 0 \le t < 1 \tag{4.46c}$$

Furthermore, if $\Gamma < e^{1/\Gamma}/(e^{1/\Gamma} - 1)$, then $f(t)$ is (strictly) monotonically $\left\{\begin{matrix}\text{increasing}\\ \text{decreasing}\end{matrix}\right\}$ for $\left\{\begin{matrix}t<\\ t>\end{matrix}\right\}$ $\Gamma\, \ell n(\Gamma[e^{1/\Gamma} - 1])$.

Remark The function $\Lambda(\Gamma)$ specified in Theorem IV.1 is obtained in the following way: From (3.28c), (3.81), and (3.82) we see that we need

$$\Lambda(\Gamma) \geq 2\Gamma(||\underset{\sim}{D}_1||^2 + \omega_K N^2 \sup_{[0,1)} ||\hat{\underset{\sim}{M}}(t)||_{L(H_+,H_-)}) \tag{4.47}$$

$$+ 8(||\underset{\sim}{D}_1||^2 + N^2 P^2_{1/\Gamma}(1)) = \Gamma\bar{\lambda}^*$$

where, by (3.70)

$$P_{1/\Gamma}(1) = \frac{K}{\Gamma} + ||\underset{\sim}{L}||_{L(H_+,H_-)} + \frac{1}{2}\sup_{[0,1)} ||\hat{\underset{\sim}{M}}(t)||_{L(H_+,H_-)} \tag{4.48}$$

However, from (2.14a), (2.14b), (4.31), (4.33), and the definition of H and H_+, it follows easily that there exist positive constants m_1, m_2 such that

$$||\underset{\sim}{L}||_{L(H_+,H_-)} \leq m_1\Gamma \;,\; \sup_{[0,1)} ||\hat{\underset{\sim}{M}}(t)||_{L(H_+,H_-)} \leq m_2\Gamma \tag{4.49}$$

(the computations needed to establish the existence of m_1, m_2 are similar the one which led to the estimate (4.8), e.g., from (4.8), with $b_o = \Gamma$, it follows easily that there exists $m_3 > 0$ such that $\sup_{[0,1)} ||\hat{\underset{\sim}{M}}_t||_{L(H_+,H_-)} \leq m_3\Gamma$).

From (4.48) and (4.49) it follows that there exist contants $n_1 > 0$, $n_2 > 0$, and $n_3 > 0$ (independent of Γ) such that

$$\Gamma\bar{\lambda}^* \leq 2(\Gamma+4)||\underset{\sim}{D}_1||^2 + n_1\Gamma^2 + \frac{n_2}{\Gamma^2} + n_3 \tag{4.50}$$

for all $\Gamma > 0$; the statement of the theorem now follows with $\Lambda(\Gamma)$ equal to the right-hand side of (4.50).

Besides the upper bound represented by (4.45) we also have, as a direct consequence of theorem III.4, the following

results concerning lower bounds for solutions of (2.1), subject to (4.42) and (4.43), where $\Phi(t)$ and $\Psi(t)$ are again determined by (4.31_1) and (4.31_2), respectively:

Theorem IV.2 Under the hypotheses which prevail in theorem IV.1 we have, for each real $\omega > 0$, and $t \in [0,1)$,

$$\int_\Omega \underset{\sim}{D}_i(\underset{\sim}{x},t)\underset{\sim}{D}_i(\underset{\sim}{x},t)d\underset{\sim}{x} + \gamma(t+t_{o,\gamma})^2 \geq \gamma t_{o,\gamma}^2 \exp[J(t;\gamma,t_{o,\gamma})] \tag{4.51}$$

where $t_{o,\gamma}$ is defined by

$$t_{o,\gamma} = \Big([(||\underset{\sim}{D}_1||^2(1 + \tfrac{4}{\Gamma}) + \omega_K N^2 \sup_{[0,1)} ||\hat{\underset{\sim}{M}}(t)||_{L(H_+,H_-)} \frac{4N^2}{\Gamma} P^2_{1/\Gamma}(1)]\frac{\Gamma}{\gamma} + \Gamma + 1\Big)(e^{-1/\Gamma} - 1), \tag{4.52}$$

with $P_{1/\Gamma}(1)$ given by (4.48);also,$J(t;\gamma,t_{o,\gamma})$ is defined by

$$J(t;\gamma,t_{o,\gamma}) = \Big(\Gamma^2\lambda^*(\gamma;t_{o,\gamma}) - \frac{2\Gamma}{t_{o,\gamma}}\Big)(1 - e^{t/\Gamma}) + \Gamma\lambda^*(\gamma;t_{o,\gamma})t \tag{4.53}$$

with

$$\lambda^*(\gamma;t_{o,\gamma}) = \frac{2}{\gamma t_{o,\gamma}^2}\Big[(1 + \tfrac{4}{\Gamma})||\underset{\sim}{D}_1||^2 + (1 + \tfrac{1}{\Gamma}(1 + t_{o,\gamma})) + 2N^2(\omega_K \sup_{[0,1)} ||\hat{\underset{\sim}{M}}(t)||_{L(H_+,H_-)} + \tfrac{4}{\Gamma} P^2_{1/\Gamma}(1))\Big] \tag{4.54}$$

Furthermore, $J(t;\gamma,t_{o,\gamma})$ is non-negative and (strictly) monotonically increasing for $0 \leq t < 1$.

Remarks We could, of course, examine the consequences of the other theorems contained in §3 as regards the growth behavior of electric displacement fields in a wide variety of holohedral isotropic dielectric materials which conform to the basic constitutive theory represented by (1.22a) and (1.22b). Clearly, examples which can be categorized as belonging to each of the cases considered in the previous section may be easily constructed by selecting suitable memory functions $\phi(t)$, $\psi(t)$ in (1.22a) and (1.22b), respectively; we leave the construction of such examples to the interested reader. In future work we shall return to consider the abstract system (2.17), (2.18) and will examine other applications to a variety of non well-posed initial boundary value problems. In particular, our work may be easily generalized to cover the case where the abstract equation has the form

$$\underset{\sim}{u}_{tt} - \underset{\sim}{K}\underset{\sim}{u}_t - \underset{\sim}{L}\underset{\sim}{u} + \int_0^t \underset{\sim}{M}(t-\tau)\underset{\sim}{u}(\tau)d\tau = \underset{\sim}{H}(t) \tag{4.55}$$

with $\underset{\sim}{K} \in L^2([0,T); \mathcal{L}(H_+,H_-))$ either positive definite or negative definite for all t, $0 \le t < T$, for some $T > 0$, and $H: [0,\infty) \to H_+$ sufficiently smooth. The abstract problem (4.55), (2.18) can then be viewed as modeling the evolution of the displacement vector in an isothermal linear viscoelastic material with nonzero past history and a time dependent (monotonically increasing or decreasing) material density.

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It is shown that the evolution of the electric displacement field in a simple class of holohedral isotropic dielectrics can be modeled by an initial value problem associated with a certain (damped) linear integrodifferential equation in Hilbert space. By employing logarithmic convexity arguments ~~we derive~~ are derived growth estimates for solutions of this integrodifferential equation which lie in uniformly bounded subsets of the appropriate Hilbert space; ~~our~~ the → next Page

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results yield both upper and lower bounds for the magnitude of the electric displacement field in the class of isotropic holohedral dielectrics which is modeled by the abstract initial-value problem.